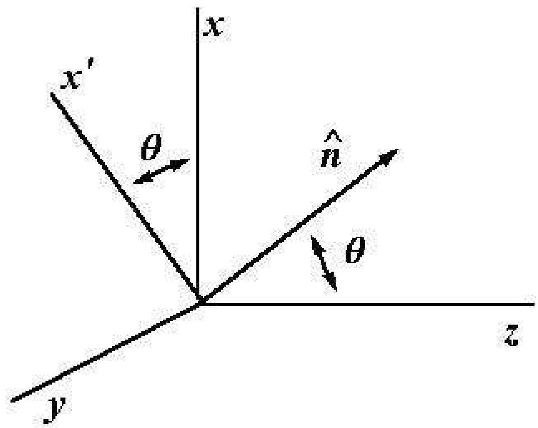
PHY 5347
Homework Set 6 Solutions – Kimel

1. 10.1
   a) Let us first simplify the expression we want to get for the cross section. Using $\hat{n}_0 = \hat{z}$,

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\hat{\varepsilon}_0 \cdot \hat{n}|^2 - \frac{1}{4}|\hat{n} \cdot (\hat{z} \times \hat{\varepsilon}_0)|^2 - \hat{z} \cdot \hat{n} \right]$$

Orienting the system as



and using

$$\hat{\varepsilon}_0 = \alpha_0 \hat{x} + \beta_0 \quad , \quad \text{with } |\alpha_0|^2 + |\beta_0|^2 = 1$$

$$\hat{n} = \cos\theta \hat{z} + \sin\theta \hat{x}$$

then

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\alpha_0|^2 \sin^2\theta - \frac{1}{4}|\beta_0|^2 \sin^2\theta - \cos\theta \right]$$

Using the result for the perfectly conducting sphere Eq. (10.14)

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}, \hat{\varepsilon}_0, \hat{n}_0) = k^4 a^6 \left| \hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 - \frac{1}{2}(\hat{z} \times \hat{\varepsilon}_0) \cdot (\hat{n} \times \hat{\varepsilon}^*) \right|^2$$

Using $\hat{\varepsilon}_\perp =$

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_\perp, \hat{\varepsilon}_0, \hat{n}_0) = k^4 a^6 \left| \beta_0 - \frac{1}{2}\beta_0 \cos\theta \right|^2 = k^4 a^6 |\beta_0|^2 \left(1 - \frac{1}{2}\cos\theta\right)^2$$

Similarly $\hat{\varepsilon}_\parallel = \hat{x}'$

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_\parallel, \hat{\varepsilon}_0, \hat{n}_0) = k^4 a^6 \left| \alpha_0 \cos\theta - \frac{1}{2}\alpha_0 \right|^2 = k^4 a^6 |\alpha_0|^2 \left(\cos\theta - \frac{1}{2}\right)^2$$

By definition

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = \frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_\perp, \hat{\varepsilon}_0, \hat{n}_0) + \frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_\parallel, \hat{\varepsilon}_0, \hat{n}_0)$$

$$= k^4 a^6 \left[|\beta_0|^2 \left(1 - \frac{1}{2}\cos\theta\right)^2 + |\alpha_0|^2 \left(\cos\theta - \frac{1}{2}\right)^2 \right]$$

which simplifies to

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\alpha_0|^2 \sin^2\theta - \frac{1}{4}|\beta_0|^2 \sin^2\theta - \cos\theta \right]$$

using $|\alpha_0|^2 + |\beta_0|^2 = 1,$ and $\cos^2\theta = 1 - \sin^2\theta.$

b) If $\hat{\varepsilon}_0$ is linearly polarized making an angle ϕ with respect to the x axis, then

$$\hat{\varepsilon}_0 = \alpha_0 \hat{x} + \beta_0 \quad = \cos\phi \hat{x} + \sin\phi \quad , \text{ so } \alpha_0 = \cos\phi, \; \beta_0 = \sin\phi$$

Then from part a)

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\alpha_0|^2 \sin^2\theta - \frac{1}{4}|\beta_0|^2 \sin^2\theta - \cos\theta \right]$$

$$= k^4 a^6 \left[\frac{5}{4} - \cos^2\phi \sin^2\theta - \frac{1}{4} \sin^2\phi \sin^2\theta - \cos\theta \right]$$

Using $\cos 2\phi = \cos^2\phi - \sin^2\phi,$ this expression simplifies to

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{8}(1 + \cos^2\theta) - \frac{3}{8} \sin^2 \cos 2\phi - \cos\theta \right]$$

as desired.