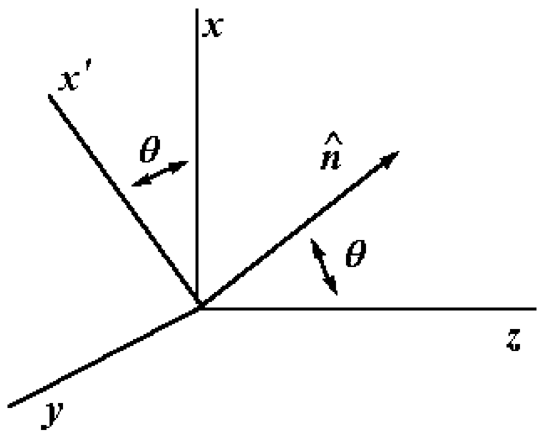
PHY 5347
Homework Set 6 Solutions – Kimel

1. 10.2

Orienting the system as



Then

$$\hat{n} = \cos\theta\hat{z} + \sin\theta\hat{x}$$

Using the result for the perfectly conducting sphere Eq. (10.14) and writing $\hat{\varepsilon}_0 = \alpha_0\hat{x} + \beta_0\hat{y}$. where $|\alpha_0|^2 + |\beta_0|^2 = 1$,

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}, \hat{\varepsilon}_0, \hat{n}_0) = k^4a^6 \left|\hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 - \frac{1}{2}(\hat{z} \times \hat{\varepsilon}_0) \cdot (\hat{n} \times \hat{\varepsilon}^*)\right|^2$$

Using $\hat{\varepsilon}_\perp = \hat{y}$

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_\perp, \hat{\varepsilon}_0, \hat{n}_0) = k^4a^6 \left|\beta_0 - \frac{1}{2}\beta_0\cos\theta\right|^2 = k^4a^6 |\beta_0|^2 \left(1 - \frac{1}{2}\cos\theta\right)^2$$

Similarly $\hat{\varepsilon}_{||} = \hat{x}'$

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_{||}, \hat{\varepsilon}_0, \hat{n}_0) = k^4a^6 \left|\alpha_0\cos\theta - \frac{1}{2}\alpha_0\right|^2 = k^4a^6 |\alpha_0|^2 \left(\cos\theta - \frac{1}{2}\right)^2$$

By definition

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = \frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_\perp, \hat{\varepsilon}_0, \hat{n}_0) + \frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_{||}, \hat{\varepsilon}_0, \hat{n}_0)$$

$$= k^4a^6 \left[|\beta_0|^2 \left(1 - \frac{1}{2}\cos\theta\right)^2 + |\alpha_0|^2 \left(\cos\theta - \frac{1}{2}\right)^2\right]$$

which simplifies to

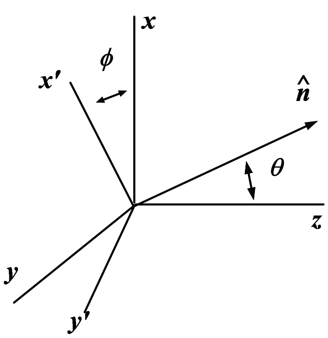
$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\alpha_0|^2 \sin^2\theta - \frac{1}{4}|\beta_0|^2 \sin^2\theta - \cos\theta\right]$$

using $|\alpha_0|^2 + |\beta_0|^2 = 1$, and $\cos^2\theta = 1 - \sin^2\theta$.

b) If $\hat{\varepsilon}_0$ is a linear combination of circular polarizations

$$\hat{\varepsilon}_0 = \frac{1}{\sqrt{1+r^2}\sqrt{2}}\left[\hat{x}' + i\hat{y}' + re^{i\alpha}(\hat{x}' - i\hat{y}')\right]$$

As is stated in problem 10.1, ϕ is measured with respect to the $\hat{x}'$ axis. For orientation, see the figure:



In term of the unit vectors in the x and y directions, respectively

$$\hat{x}' = \cos\phi\hat{x} + \sin\phi\hat{y};\ \hat{y}' = \cos\phi\hat{y} - \sin\phi\hat{x}$$

corresponding to a rotation about the z axis of ϕ.Thus

$$\alpha_0 = \frac{1}{\sqrt{1+r^2}\sqrt{2}}\left[\cos\phi\left(1+re^{i\alpha}\right) - i\sin\phi\left(1-re^{i\alpha}\right)\right]$$

$$\beta_0 = \frac{1}{\sqrt{1+r^2}\sqrt{2}}\left[\sin\phi\left(1+re^{i\alpha}\right) + i\cos\phi\left(1-re^{i\alpha}\right)\right]$$

Notice that

$$|\alpha_0|^2 = \frac{1}{(1+r^2)\,2}\left[\cos^2\phi\left(1+r^2+2r\cos\alpha\right) + \sin^2\phi(1+r^2-2r\cos\alpha) - 4r\sin\phi\cos\phi\sin\alpha\right]$$

$$|\beta_0|^2 = \frac{1}{(1+r^2)\,2}\left[\sin^2\phi\left(1+r^2+2r\cos\alpha\right) + \cos^2\phi(1+r^2-2r\cos\alpha) + 4r\sin\phi\cos\phi\sin\alpha\right]$$

and

$$|\alpha_0|^2 + |\beta_0|^2 = 1$$

Plugging these results into

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\alpha_0|^2 \sin^2\theta - \frac{1}{4}|\beta_0|^2 \sin^2\theta - \cos\theta\right]$$

gives for the terms not linear with r,

$$\frac{d\sigma_1}{d\Omega} = k^4 a^6 \left[\frac{5}{8}\left(1 + \cos^2\theta\right) - \cos\theta\right]$$

whereas the terms linear in r contribute

$$\frac{d\sigma_2}{d\Omega} = k^4 a^6 \left[-\frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi - \alpha)\right]$$

where I have used

$$\cos(2\phi - \alpha) = \cos 2\phi \cos\alpha + \sin 2\phi \sin\alpha$$

and

$$\cos 2\phi = \cos^2\phi - \sin^2\phi$$

Adding the two contributions gives

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{8}\left(1 + \cos^2\theta\right) - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi - \alpha)\right]$$

the desired result.