PHY 5347
Homework Set 8 Solutions – Kimel

4. 11.6
Background:

$$dt = \gamma(\tau)d\tau, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

where dt is measured in the K_0 frame and $d\tau$ is the proper time. Using the Lorentz transformation for acceleration

$$\vec{a}_{||} = \frac{(1-\frac{v^2}{c^2})^{3/2}}{(1-\frac{\vec{v}\cdot\vec{u}'}{c^2})}\vec{a}'_{||}, \text{ but in this case } \vec{u}' = 0$$

$$a = \frac{dv}{dt} = (1-v^2/c^2)^{3/2}\frac{dv'}{d\tau}, \text{ where } a_0 \text{ is acceleration in } K_0 \text{ and } a' = \frac{dv'}{d\tau}$$

So

$$dv = (1-v^2/c^2)^{3/2}a'dt = (1-v^2/c^2)a'd\tau$$

$$\int_0^v \frac{dv}{(1-v^2/c^2)} = a'\int_0^\tau d\tau$$

$$-\frac{1}{2}c\ln(c-v) + \frac{1}{2}c\ln(c+v) = a'\tau = \frac{1}{2}c\ln\left(\frac{c+v}{c-v}\right)$$

$$e^{\frac{2a'\tau}{c}} = \left(\frac{c+v}{c-v}\right) \rightarrow v(\tau) = c\frac{e^{\frac{2}{c}a'\tau}-1}{1+e^{\frac{2}{c}a'\tau}} = c\left(\frac{e^{\frac{1}{c}a'\tau}-e^{-\frac{1}{c}a'\tau}}{e^{\frac{1}{c}a'\tau}+e^{-\frac{1}{c}a'\tau}}\right) = c\tanh(\frac{a'\tau}{c})$$

$$\beta(\tau) = \tanh(\frac{a'\tau}{c})$$

$$\frac{dx}{dt} = v(t) \rightarrow dx = v(\tau)\gamma(\tau)d\tau$$

Or

$$x_{12} \equiv \int dx = \int_{\tau_1}^{\tau_2} v(\tau)\gamma(\tau)\,d\tau = c\int_{\tau_1}^{\tau_2}\frac{\tanh(\frac{a'\tau}{c})}{\sqrt{1-\tanh^2(\frac{a'\tau}{c})}}d\tau = c\int_{\tau_1}^{\tau_2}\tanh(\frac{a'\tau}{c})\cosh(\frac{a'\tau}{c})d\tau$$

$$= c\int_{\tau_1}^{\tau_2}\left(\sinh\frac{1}{c}a'\tau\right)d\tau = \frac{c^2}{a}\cosh(\frac{a'\tau}{c})|_{\tau_1}^{\tau_2}$$

Let's work part b) first:

b) The 10–year time frame going out is divided into two parts:

1^{st} 5 years: $\tau_1 = 0$, $\tau_2 = 5$ yrs, $a' = g$

$$x_{02} = \frac{c^2}{a'}\left[\cosh(\frac{g\tau_2}{c}) - 1\right] = c\left[\tau_2\left(\frac{c}{a'\tau_2}\right)\left(\cosh(\frac{a'\tau_2}{c}) - 1\right)\right]$$

$$\frac{a'\tau_2}{c} = \frac{9.81 \cdot 5 \cdot 365 \cdot 24 \cdot 3600}{3 \times 10^8} = 5.16$$

$$= c\left[\tau_2\left(\frac{c}{a'\tau_2}\right)\left(\cosh(\frac{a'\tau_2}{c}) - 1\right)\right] = \mathrm{c} \cdot 5 \cdot \mathrm{yrs}\frac{1}{5.16}\left[\cosh(5.16) - 1\right]$$

$$= 83.4 \text{ light-years}$$

2^{nd} 5 years: $\tau_1 = 5$ yrs, $\tau_2 = 10$ yrs, $a' = -g$
By symmetry, this is the same as the first five years, 83.4 light-years.

Total distance after 10 years:

$$x_{Total} = (83.4 + 83.4) \text{ light-years} = 166.8 \text{ light-years}$$

a) Working out the time that elapses in the K_0 frame.

1^{st}5 years:

$$dt = \gamma d\tau \rightarrow t = \int_0^\tau \frac{1}{\sqrt{1 - \tanh^2(a'\tau/c)}} d\tau = \int_0^\tau \cosh(a'\tau/c) d\tau$$

$$= \frac{c}{a'}\sinh(\frac{a'\tau}{c}) = \tau\left(\frac{1}{a'\tau/c}\right)\sinh(\frac{a'\tau}{c})$$

$$\rightarrow t = 5\mathrm{yrs} \cdot \frac{1}{5.16}\sinh(5.16) = 84.4 \text{ yrs}$$

2^{nd} 5 years:

$$t = 84.4 \text{ yrs by symmetry}$$

By symmetry, the return trip takes as long as the trip out.

$$\rightarrow t_{tot} = 2 \cdot (84.4 + 84.4) \text{ yrs} = 337.6 \text{ yrs}$$

It is the year $2100 + 337 = 2437$ on earth and the twin on earth is 357 yrs old!