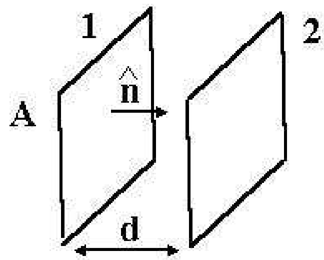
PHY 5346
HW Set 2 Solutions – Kimel

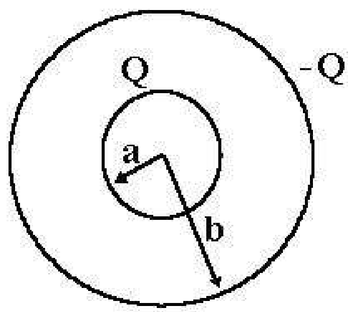
2. 1.8 We will be using Gauss's law $\int \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$
a) 1) Parallel plate capacitor



From Gauss's law $E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0} = \frac{\phi_{12}}{d} \to Q = \frac{A\varepsilon_0\phi_{12}}{d}$

$$W = \frac{\varepsilon_0}{2}\int E^2 d^3x = \frac{\varepsilon_0 E^2 Ad}{2} = \frac{\varepsilon_0\left(\frac{Q}{A\varepsilon_0}\right)^2 Ad}{2} = \frac{1}{2\varepsilon_0}\frac{Q^2}{A}d = \frac{1}{2\varepsilon_0}\frac{\left(\frac{A\varepsilon_0\phi_{12}}{d}\right)^2}{A}d = \frac{1}{2}\varepsilon_0 A\frac{\phi_{12}^2}{d}$$

2) Spherical capacitor



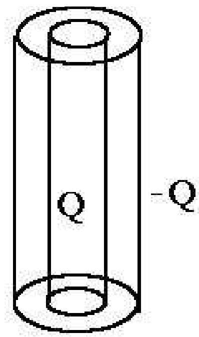
From Gauss's law, $E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2},\ a < r < b.$

$$\phi_{12} = \int_a^b E dr = \frac{Q}{4\pi\varepsilon_0}\int_a^b r^{-2} dr = \frac{1}{4}\frac{Q}{\pi\varepsilon_0}\frac{(b-a)}{ba} \to Q = \frac{4\pi\varepsilon_0 ba\phi_{12}}{(b-a)}$$

$$W = \frac{\varepsilon_0}{2}\int E^2 d^3x = \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0}\right)^2\int_a^b 4\pi\frac{r^2 dr}{r^4} = \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0}\right)^2\left(-4\pi\frac{-b+a}{ba}\right) = \frac{1}{8\varepsilon_0}\frac{Q^2}{\pi}\frac{(b-a)}{ba}$$

$$W = \frac{1}{8\varepsilon_0}\frac{\left(\frac{4\pi\varepsilon_0 ba\phi_{12}}{(b-a)}\right)^2}{\pi}\frac{(b-a)}{ba} = 2\pi\varepsilon_0 ba\frac{\phi_{12}^2}{(b-a)}$$

3) Cylindrical conductor



From Gauss's law, $E 2\pi r L = \frac{\lambda L}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$

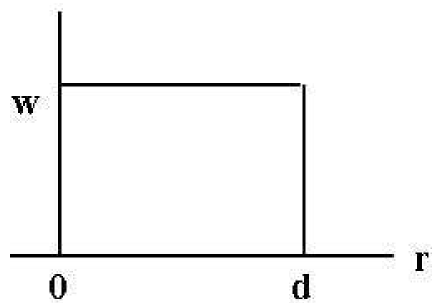
$$\phi_{12} = \int_a^b E dr = \frac{Q}{2\pi\varepsilon_0 L}\int_a^b \frac{dr}{r} = \frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right)$$

$$W = \frac{\varepsilon_0}{2}\int E^2 d^3x = \frac{\varepsilon_0}{2}\left(\frac{Q}{2\pi\varepsilon_0 L}\right)^2 2\pi L\int_a^b \frac{rdr}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{L}\ln\left(\frac{b}{a}\right)$$

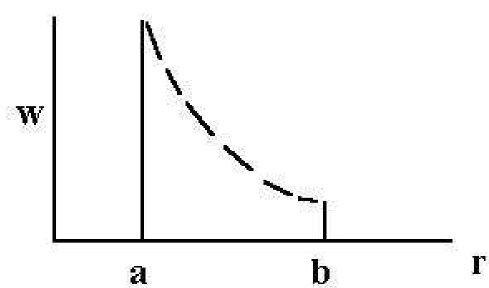
$$W = \frac{1}{4\pi\varepsilon_0}\frac{\left(\frac{2\pi\varepsilon_0 L\phi_{12}}{\ln\left(\frac{b}{a}\right)}\right)^2}{L}\ln\left(\frac{b}{a}\right) = \pi\varepsilon_0 L\frac{\phi_{12}^2}{\ln\frac{b}{a}}$$

b) $w = \frac{\varepsilon_0}{2}E^2$

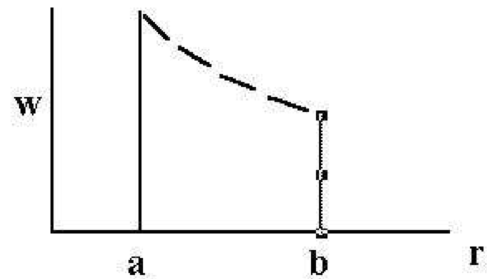
1) $w(r) = \frac{\varepsilon_0}{2}\left(\frac{Q}{A\varepsilon_0}\right)^2 = \frac{1}{2\varepsilon_0}\frac{Q^2}{A^2}$ $0 < r < d$, $= 0$ otherwise.



2) $w(r) = \frac{\varepsilon_0}{2}\left(\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\right)^2 = \frac{1}{32\varepsilon_0\pi^2}\frac{Q^2}{r^4}$, $a < r < b$, $= 0$, otherwise



3) $w(r) = \frac{\varepsilon_0}{2}\left(\frac{Q}{2\pi\varepsilon_0 L r}\right)^2 = \frac{1}{8\varepsilon_0}\frac{Q^2}{\pi^2 L^2 r^2}$, $= 0$ otherwise.


w
a
b
r