PHY 5346
HW Set 4 Solutions – Kimel

2. 2.22

a) Using the fact that for the interior problem, the normal derivative is outward, rather than inward, we have the potenial given by the negative of Eq. (2.21), which takes the form, when $\theta = 0$ and $\gamma = \theta'$

$$\Phi(z) = -\frac{Va\left(z^2 - a^2\right)2\pi}{4\pi}\int_0^1 \left(\frac{1}{\left(a^2 + z^2 - 2azx\right)^{3/2}} - \frac{1}{\left(a^2 + z^2 + 2azx\right)^{3/2}}\right) dx$$

where I've replaced $\cos\theta'$ by x in the integral. The integral yields

$$\Phi(z) = \frac{Va}{z}\left(1 - \frac{\left(a^2 - z^2\right)}{a\sqrt{a^2 + z^2}}\right)$$

$$\Phi(z) = \frac{Va}{z}\left[\frac{3}{2}\frac{z^2}{a^2} + \left(-\frac{7}{8}\right)\frac{z^4}{a^4} + O\left(z^6\right)\right]$$

which agrees with Eq. (2.27) if $\cos\theta = 1$.

b) For $z > a$, we have, using Eq. (2.22)

$$E_z = -\frac{\partial}{\partial z}V\left(1 - \frac{\left(z^2 - a^2\right)}{z\sqrt{a^2 + z^2}}\right) = E_z(z) = \frac{Va^2}{\left(a^2 + z^2\right)^{\frac{3}{2}}}\left(3 + \frac{a^2}{z^2}\right)$$

For $|z| < a$,

$$E_z(z) = -\frac{\partial}{\partial z}\frac{Va}{z}\left(1 - \frac{\left(a^2 - z^2\right)}{a\sqrt{a^2 + z^2}}\right) = E_z(z) = -\frac{V}{a}\left(-\frac{a^2}{z^2} + \frac{3a^3 + a^5/z^2}{\left(a^2 + z^2\right)^{\frac{3}{2}}}\right)$$

in agreement with the book. Expanding the second form in a Taylor series exapansion about $z = 0$ gives

$$E_z = -\frac{3}{2a}V + \frac{21}{8}\frac{V}{a^3}z^2 - \frac{55}{16}\frac{V}{a^5}z^4 + O\left(z^6\right)$$

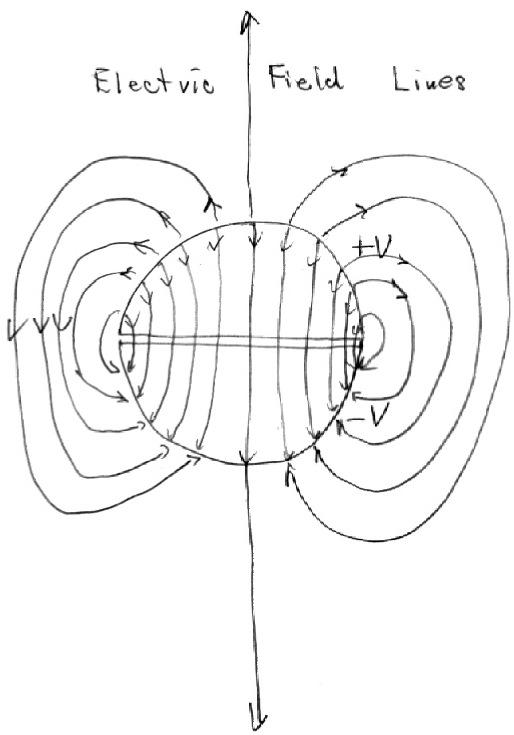
which shows $E_z(0) = -\frac{3}{2a}V$, as required. Also, from the second form

$$E_z(a) = -\frac{V}{a}\left(-1 + \sqrt{2}\right)$$

From the first form, on the outside, we get

$$E_z(a) = \frac{\sqrt{2}V}{a}$$

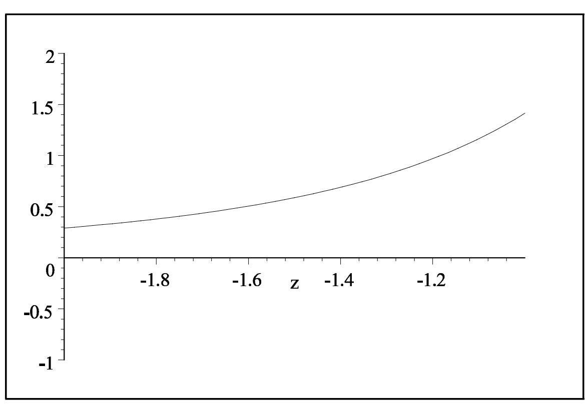
c) First look at a plot of the field lines:



Next, look at $E(z)$ in the region (-$2a$, $2a$). I will make the plot in units of a.

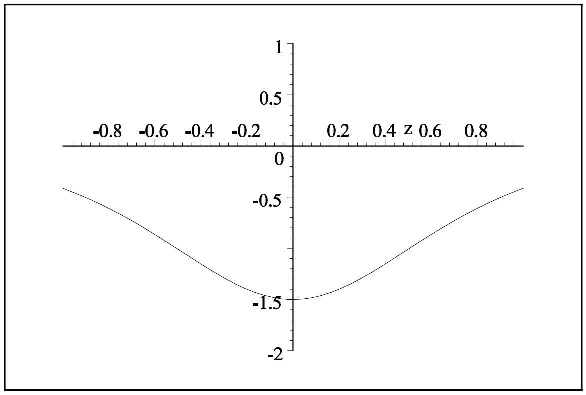
$$E(z) = \frac{1}{(1+z^2)^{\frac{3}{2}}}\left(3 + \frac{1}{z^2}\right)$$

$E(z)$



$$E(z) = -\left(-\frac{1}{z^2} + \frac{3 + 1/z^2}{(1+z^2)^{\frac{3}{2}}}\right)$$

$E(z)$



$$E(z) = \frac{1}{(1+z^2)^{\frac{3}{2}}}\left(3 + \frac{1}{z^2}\right)$$

$E(z)$