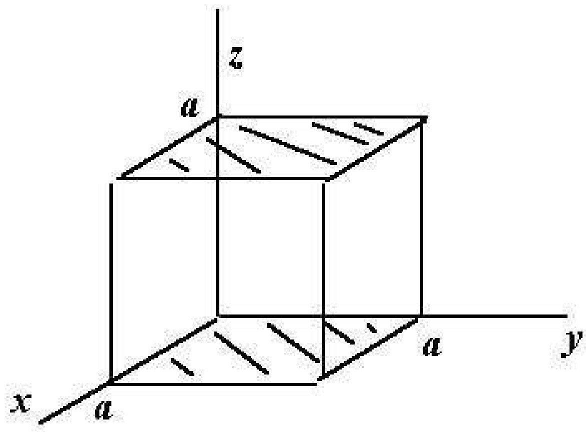
PHY 5346
HW Set 5 Solutions – Kimel

2. 2.23 The system is pictured in the following figure:



a) As suggested in the text and in class, we will superpose solutions of the form (2.56) for the two sides with $V(x,y,z) = V$.

1) First consider the side $V(x,y,a) = z$:

$$\Phi_1(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z)$$

with $\alpha_n = \frac{n\pi}{a},\ \beta_m = \frac{m\pi}{a},\ \gamma_{nm} = \frac{\pi}{a}\sqrt{n^2+m^2}$. Projecting out A_{nm} using the orthogonality of the sine functions,

$$A_{nm} = \frac{16V}{\sinh(\gamma_{nm}a) nm\pi^2}$$

where both $n,$ and m are odd. (Later we will use $n = 2p+1,\ m = 2q+1$)

2) In order to express $\Phi_2(x,y,z)$ in a form like the above, we make the coordinate transformation

$$x' = y,\ y' = x,\ z' = -z+a$$

So

$$\Phi_2(x,y,z) = \Phi_1(x',y',z') = \Phi_1(y,x,= z+a)$$

$$\Phi(x,y,z) = \Phi_1(x,y,z) + \Phi_2(x,y,z)$$

b)

$$\Phi\left(\frac{a}{2},\frac{a}{2},\frac{a}{2}\right) = \frac{16 \cdot V}{\pi^2} \sum_{p,q=0}^{\infty} \frac{(-1)^{p+q}}{(2p+1)(2q+1)\cosh\left(\gamma_{nm}\frac{a}{2}\right)}$$

where I have used the identity

$$\sinh(\gamma_{nm}a) = 2\sinh\left(\frac{\gamma_{nm}a}{2}\right)\cosh\left(\frac{\gamma_{nm}a}{2}\right)$$

Let $f(p,q) \equiv \sum_{p,q=0}^{\infty} \frac{(-1)^{p+q}}{(2p+1)(2q+1)\cosh\left(\sqrt{(2p+1)^2+(2q+1)^2}\,\frac{\pi}{2}\right)}$

(p,q)	$f(p,q)$	Error	Sum
0,0	0.213484	4.4%	.214384
1,0	-0.004641	2.13%	0.20974
0,1	-0.004641	0.013%	0.20510
1,1	0.0002835	0.015%	0.20539

The first three terms give an accuracy of 3 significant figures.

$$\Phi\left(\frac{a}{2},\frac{a}{2},\frac{a}{2}\right) = \frac{16\cdot 0.20539}{\pi^2}V = 0.33296V$$

$$\Phi_{av}\left(\frac{a}{2},\frac{a}{2},\frac{a}{2}\right) = \frac{2}{6}V = 0.333\ldots V$$

c)

$$\sigma(x,y,a) = -\varepsilon_0\frac{\partial}{\partial z}\Phi|_{z=a}$$

$$\sigma(x,y,a) = -\frac{16\varepsilon_0}{\pi^2}V$$

$$= -\frac{16\varepsilon_0}{\pi^2}V\sum_{n,m\text{ odd}}^{\infty}\sin(\alpha_n x)\sin(\beta_m y)\left[\frac{(\cosh(\gamma_{nm}a)-1}{\sinh(\gamma_{nm}a)}\right]$$

$$\sigma(x,y,a) = -\frac{16\varepsilon_0}{\pi^2}V\sum_{n,m\text{ odd}}^{\infty}\sin(\alpha_n x)\sin(\beta_m y)\tanh\left(\frac{\gamma_{nm}a}{2}\right)$$