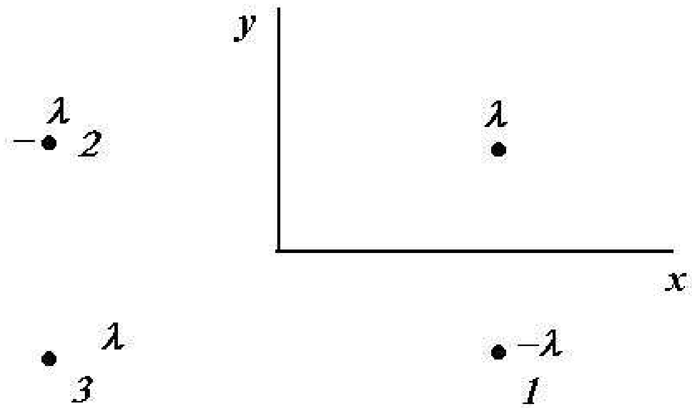
PHY 5346
HW Set 3 Solutions – Kimel

2. 2.3 The system is described by



a) Given the potenial for a line charge in the problem, we write down the solution from the figure,

$$\phi_T = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\frac{R^2}{(\vec{x}-\vec{x}_o)^2} - \ln\frac{R^2}{(\vec{x}-\vec{x}_{o1})^2} - \ln\frac{R^2}{(\vec{x}-\vec{x}_{o2})^2} + \ln\frac{R^2}{(\vec{x}-\vec{x}_{o3})^2}\right]$$

Looking at the figure when $y = 0,\ (\vec{x}-\vec{x}_o)^2 = (\vec{x}-\vec{x}_{o1})^2,\ (\vec{x}-\vec{x}_{o2})^2 = (\vec{x}-\vec{x}_{o3})^2$, so $\phi_T|_{y=0} = 0$
Similarly, when $x = 0,\ (\vec{x}-\vec{x}_o)^2 = (\vec{x}-\vec{x}_{o2})^2,\ (\vec{x}-\vec{x}_{o1})^2 = (\vec{x}-\vec{x}_{o3})^2,$ so $\phi_T|_{x=0} = 0$
On the surface $\phi_T = 0,$ so $\delta\phi_T = 0,$ however,

$$\delta\phi_T = \frac{\partial\phi_T}{\partial x_t}\delta x_t = 0 \rightarrow \frac{\partial\phi_T}{\partial x_t} = 0 \rightarrow E_t = 0$$

b) We remember

$$\sigma = -\varepsilon_0\frac{\partial\phi_T}{\partial y} = \frac{-\lambda}{\pi}\left[\frac{y_0}{(x-x_0)^2+y_0^2} - \frac{y_0}{(x+x_0)^2+y_0^2}\right]$$

where I've applied the symmetries derived in a). Let

$$\sigma/\lambda = \frac{-1}{\pi}\left[\frac{y_0}{(x-x_0)^2+y_0^2} - \frac{y_0}{(x+x_0)^2+y_0^2}\right]$$

This is an easy function to plot for various combinations of the position of the original line charge (x_0, y_0).

c) If we integrate over a strip of width $\Delta z,$ we find, where we use the integral

$$\int_0^\infty \frac{1}{(x\mp x_0)^2+y_0^2}dx = \frac{1}{2}\frac{\pi \pm 2\arctan\frac{x_0}{y_0}}{y_0}$$

$$\Delta Q = \int_0^\infty \sigma dx\Delta z \rightarrow \frac{\Delta Q}{\Delta z} = \int_0^\infty \sigma dx = \frac{-2\lambda}{\pi}\tan^{-1}\left(\frac{x_0}{y_0}\right)$$

and the total charge induced on the plane is $-\infty,$ as expected.
d)
Expanding

$$\ln\left(\frac{R^2}{(x-x_0)^2+(y-y_0)^2}\right)-\ln\left(\frac{R^2}{(x-x_0)^2+(y+y_0)^2}\right)-\ln\left(\frac{R^2}{(x+x_0)^2+(y-y_0)^2}\right)+\ln\left(\frac{R^2}{(x+x_0)^2+(y+\dots}\right.$$

to lowest non-vanishing order in $x_0,\ y_0$ gives

$$16\frac{xy}{(x^2+y^2)^2}y_0x_0$$

so

$$\phi\to\phi_{asym}=\frac{4\lambda}{\pi\varepsilon_0}\frac{xy}{(x^2+y^2)^2}y_0x_0$$

This is the quadrupole contribution.