PHY 5346
HW Set 3 Solutions – Kimel

3. 2.5
a)

$$W = \int_r^\infty |F| dy = \frac{q^2 a}{4\pi\varepsilon_0} \int_r^\infty \frac{dy}{y^3\left(1 - \frac{a^2}{y^2}\right)^2} = \frac{q^2 a}{8\pi\varepsilon_0 (r^2 - a^2)}$$

Let us compare this to disassemble the charges

$$-W' = -\frac{1}{8\pi\varepsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} = \frac{1}{4\pi\varepsilon_0}\left[\frac{aq^2}{r} \frac{1}{r\left(1 - \frac{a^2}{r^2}\right)}\right] = \frac{q^2 a}{4\pi\varepsilon_0 (r^2 - a^2)} > W$$

The reason for this difference is that in the first expression W, the image charge is moving and changing size, whereas in the second, whereas in the second, they don't.

b) In this case

$$W = \int_r^\infty |F| dy = \frac{q}{4\pi\varepsilon_0}\left[\int_r^\infty \frac{Q dy}{y^2} - qa^3 \int_r^\infty \frac{(2y^2 - a^2)}{y(y^2 - a^2)^2} dy\right]$$

Using standard integrals, this gives

$$W = \frac{1}{4\pi\varepsilon_0}\left[\frac{q^2 a}{2(r^2 - a^2)} - \frac{q^2 a}{2r^2} - \frac{qQ}{r}\right]$$

On the other hand

$$-W' = \frac{1}{4\pi\varepsilon_0}\left[\frac{aq^2}{(r^2 - a^2)} - \frac{q(Q + \frac{a}{r}q)}{r}\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{aq^2}{(r^2 - a^2)} - \frac{q^2 a}{r^2} - \frac{qQ}{r}\right]$$

The first two terms are larger than those found in W for the same reason as found in a), whereas the last term is the same, because Q is fixed on the sphere.