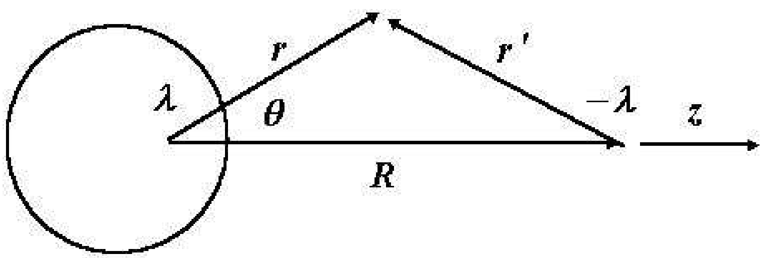
PHY 5346
HW Set 4 Solutions – Kimel

1. 2.8 The system is pictured below



a) Using the known potential for a line charge, the two line charges above give the potential

$$\phi(\vec{r}) = \frac{1}{2\pi\varepsilon_0}\lambda \ln\frac{r'}{r} = V, \text{ a constant. Let us define } V' = 4\pi\varepsilon_0 V$$

Then the above equation can be written

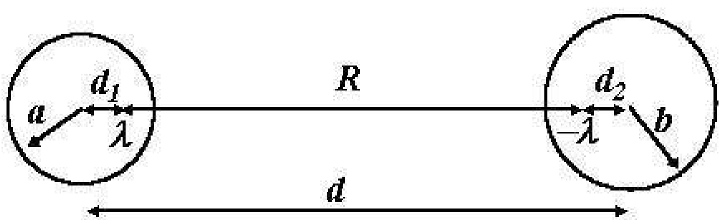
$$\left(\frac{r'}{r}\right)^2 = e^{\frac{V'}{\lambda}} \text{ or } r'^2 = r^2 e^{\frac{V'}{\lambda}}$$

Writing $r'^2 = \left(\vec{r} - \vec{R}\right)^2$, the above can be written

$$\left(\vec{r} + \hat{z}\frac{R}{(e^{\frac{V'}{\lambda}} - 1)}\right)^2 = \frac{R^2 e^{\frac{V'}{\lambda}}}{(e^{\frac{V'}{\lambda}} - 1)^2}$$

The equation is that of a circle whose center is at $-\hat{z}\frac{R}{e^{\frac{V'}{\lambda}}-1}$, and whose radius is $a = \frac{Re^{\frac{V'}{2\lambda}}}{(e^{\frac{V'}{\lambda}}-1)}$

b) The geometry of the system is shown in the fugure.



Note that

$$d = R + d_1 + d_2$$

with

$$d_1 = \frac{R}{e^{\frac{V_a'}{\lambda}}-1}, \quad d_2 = \frac{R}{e^{\frac{-V_b'}{\lambda}}-1}$$

and

$$a = \frac{Re^{\frac{V_a'}{2\lambda}}}{(e^{\frac{V_a'}{\lambda}} - 1)}, \quad b = \frac{Re^{\frac{-V_b'}{2\lambda}}}{(e^{\frac{-V_b'}{\lambda}} - 1)}$$

Forming

$$d^2 - a^2 - b^2 = \left(R + \frac{R}{e^{\frac{V_a'}{\lambda}-1}} + \frac{R}{e^{\frac{-V_b'}{\lambda}-1}} \right)^2 - \left(\frac{Re^{\frac{V_a'}{2\lambda}}}{(e^{\frac{V_a'}{\lambda}} - 1)} \right)^2 - \left(\frac{Re^{\frac{-V_b'}{2\lambda}}}{(e^{\frac{-V_b'}{\lambda}} - 1)} \right)^2$$

or

$$d^2 - a^2 - b^2 = \frac{R^2\left(e^{\frac{V_a'-V_b'}{\lambda}} + 1 \right)}{(e^{\frac{V_a'}{\lambda}} - 1)(e^{\frac{-V_b'}{\lambda}} - 1)}$$

Thus we can write

$$\frac{d^2 - a^2 - b^2}{2ab} = \frac{\left(e^{\frac{V_a'-V_b'}{\lambda}} + 1 \right)}{2e^{\frac{V_a'}{2\lambda}} e^{\frac{-V_b'}{2\lambda}}} = \frac{e^{\frac{V_a'-V_b'}{2\lambda}} + e^{\frac{-(V_a'-V_b')}{2\lambda}}}{2} = \cosh\left(\frac{V_a' - V_b'}{2\lambda} \right)$$

or

$$\frac{V_a - V_b}{\lambda} = \frac{1}{2\pi\varepsilon_0} \cosh^{-1}\left(\frac{d^2 - a^2 - b^2}{2ab} \right)$$

$$\text{Capacitance/unit length} = \frac{C}{L} = \frac{Q/L}{V_a - V_b} = \frac{\lambda}{V_a - V_b} = \frac{2\pi\varepsilon_0}{\cosh^{-1}\left(\frac{d^2-a^2-b^2}{2ab} \right)}$$

c) Suppose $a^2 << d^2$, and $b^2 << d^2$, and $a' = \sqrt{ab}$, then

$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\cosh^{-1}\left(\frac{d^2-a^2-b^2}{2a'^2} \right)} = \frac{2\pi\varepsilon_0}{\cosh^{-1}\left(\frac{d^2(1-(a^2+b^2)/d^2)}{2a'^2} \right)}$$

$$\cosh^{-1}\left(\frac{d^2(1 - (a^2 + b^2)/d^2)}{2a'^2} \right) = \frac{2\pi\varepsilon_0 L}{C}$$

$$\left(\frac{d^2(1 - (a^2 + b^2)/d^2)}{2a'^2} \right) = \frac{e^{\frac{2\pi\varepsilon_0 L}{C}}}{2} + \text{ negligible terms if } \frac{2\pi\varepsilon_0 L}{C} >> 1$$

or

$$\ln\left(\frac{d^2(1 - (a^2 + b^2)/d^2)}{a'^2} \right) = \frac{2\pi\varepsilon_0 L}{C}$$

or

$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{d^2(1-(a^2+b^2)/d^2)}{a'^2} \right)}$$

Let us defind $\alpha^2 = (a^2 + b^2)/d^2$, then

$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{d^2(1-\alpha^2)}{a'^2} \right)} = 2\pi\frac{\varepsilon_0}{\ln\frac{d^2}{a'^2}} + 2\pi\frac{\varepsilon_0}{\ln^2\frac{d^2}{a'^2}}\alpha^2 + O(\alpha^4)$$

The first term of this result agree with problem 1.7, and the second term gives the appropriate correction asked for.

d) In this case, we must take the opposite sign for $d^2 - a^2 - b^2$, since $a^2 + b^2 > d^2$. Thus

$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\cosh^{-1}\left(\frac{a^2+b^2-d^2}{2a'^2} \right)}$$

If we use the identiy, $\ln(x + \sqrt{x^2 - 1}) = \cosh^{-1}(x)$, G.&R., p. 50., then for $d = 0$

$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{a^2+b^2}{2ab} + \frac{a^2-b^2}{2ab}\right)} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{a}{b}\right)}$$

in agreement with problem 1.6.