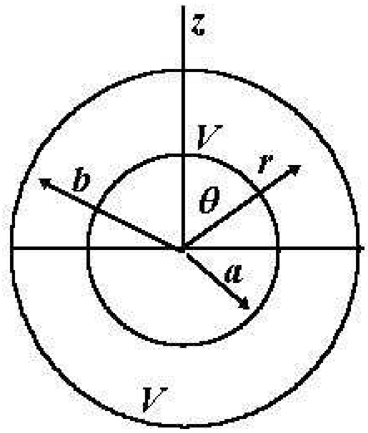
PHY 5346
HW Set 5 Solutions – Kimel

3. 3.1 The system is pictured in the following figure:



$\int_0^1 P_l(x)dx$

The problem is symmetric around the z axis so

$$\phi(r,\theta) = \sum_l (A_l r^l + B_l r^{-l-1}) P_l(\cos\theta)$$

The A_l and B_l are determined by the conditions

1)

$$\int_{-1}^{1} \phi(a,x) P_l(x) dx = \frac{2}{2l+1}(A_l a^l + B_l a^{-l-1})$$

2)

$$\int_{-1}^{1} \phi(b,x) P_l(x) dx = \frac{2}{2l+1}(A_l b^l + B_l b^{-l-1})$$

Solving these two equations gives

$$A_l = \frac{2l+1}{2(a^{2l+1} - b^{2l+1})}\left[a^{l+1} \int_{-1}^{1} \phi(a,x) P_l(x) dx - b^{l+1} \int_{-1}^{1} \phi(b,x) P_l(x) dx \right]$$

$$B_l = a^{l+1} \frac{2l+1}{2} \int_{-1}^{1} \phi(a,x) P_l(x) dx - A_l a^{2l+1}$$

Using

$$\int_{-1}^{1} \phi(a,x) P_l(x) dx = V \int_0^1 P_l(x) dx$$

$$\int_{-1}^{1} \phi(b,x) P_l(x) dx = V \int_{-1}^{0} P_l(x) dx = V(-1)^l \int_0^1 P_l(x) dx$$

So

$$A_l = \frac{2l+1}{2(a^{2l+1} - b^{2l+1})} V[a^{l+1} - b^{l+1}(-1)^l] \int_0^1 P_l(x) dx$$

$$B_l = a^{l+1} \frac{2l+1}{2} V \int_0^1 P_l(x) dx - A_l a^{2l+1}$$

Note that

$$\int_0^1 P_l(x)dx = \frac{1}{2}\int_{-1}^1 P_l(x)dx$$

for l even. For even $l > 0$, $\int_0^1 P_l(x)dx = 0$. Thus we have

$$\int_0^1 P_0(x)dx = 1;\ \int_0^1 P_1(x)dx = \frac{1}{2},\ \int_0^1 P_3(x)dx = -\frac{1}{8}$$

and

$$A_0 = \frac{V}{2},\ A_1 = \frac{3}{4(a^3-b^3)}V(a^2+b^2),\ A_3 = -\frac{7}{16(a^7-b^7)}V(a^4+b^4)$$

$$B_0 = \frac{1}{2}Va - \frac{1}{2}Va = 0$$

$$B_1 = \frac{3}{4}a^2V - \frac{3}{4(a^3-b^3)}V(a^2+b^2)a^3 = \frac{3}{4}Va^2b^2\frac{b+a}{-a^3+b^3}$$

$$B_3 = -\frac{7}{16}a^4V - a^7\left(-\frac{7}{16(a^7-b^7)}V(a^4+b^4)\right) = -\frac{7}{16}Va^4b^4\frac{b^3+a^3}{-a^7+b^7}$$

As $b \to \infty$, only the B_l terms (and A_0) survive. Thus using the general expression for $\int_0^1 P_l(x)dx$ given by (3.26)

$$\phi(r,\theta) = \frac{V}{2}\left[P_0(x) + \frac{3}{2}\frac{a^3}{r^2}P_1(x) - \frac{7}{8}\frac{a^4}{r^4}P_3(x) + \ldots\ldots\right]$$

Let's now solve the problem neglecting the outer sphere (since $b \to \infty$) using the Green's function result

(2.19) this integral give, for $\cos\theta = 1$

$$\phi(r,\theta) = \frac{V}{2}(1-\rho^2)\left[\frac{1}{(1-\rho)} - \frac{1}{\sqrt{(1+\rho^2)}}\right]$$

with $\rho = a/r$. Expanding the above,

$$\phi(r,\theta) = \frac{V}{2}\left[\frac{a}{r} + \frac{3}{2}\frac{a^2}{r^2} - \frac{7}{8}\frac{a^4}{r^4} + \ldots\ldots\right]$$

Comparing with our previous solution with $x = 1$, we see the Green's function solution differs by having a B_0 term and by not having an A_0 term. All the other higher power terms agree in the series. This difference is due to having a potential at ∞ in the original problem.