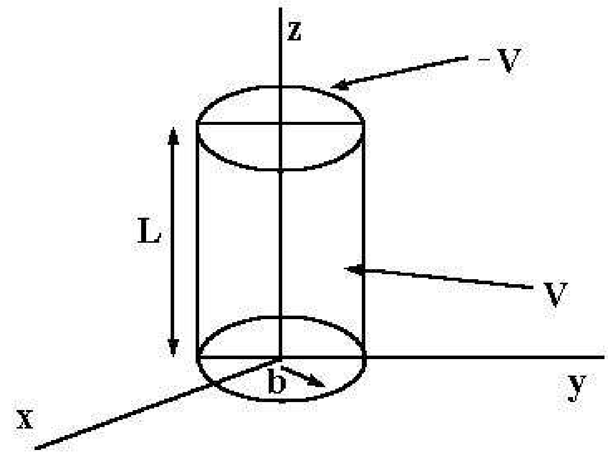
PHY 5346
HW Set 6 Solutions – Kimel

2. 3.10 This problem is described by



a) From the class notes

$$\Phi(\rho,z,\phi) = \sum_{n\nu}(A_{n\nu}\sin\nu\phi + B_{n\nu}\cos\nu\phi)I_\nu\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)$$

where

$$A_{n\nu} = \frac{2}{\pi L}\frac{1}{I_\nu\left(\frac{n\pi b}{L}\right)}\int_0^L\int_0^{2\pi} V(\phi,z)\sin(\frac{n\pi a}{L})\sin(\nu\phi)d\phi dz, \qquad \nu > 0$$

$$B_{n\nu} = \frac{2}{\pi L}\frac{1}{I_\nu\left(\frac{n\pi b}{L}\right)}\int_0^L\int_0^{2\pi} V(\phi,z)\sin(\frac{n\pi a}{L})\cos(\nu\phi)d\phi dz, \qquad \nu \neq 0$$

$$B_{n\nu} = \frac{1}{\pi L}\frac{1}{I_\nu\left(\frac{n\pi b}{L}\right)}\int_0^L\int_0^{2\pi} V(\phi,z)\sin(\frac{n\pi a}{L})d\phi dz, \qquad \nu = 0$$

Noting

$$\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin\nu\phi d\phi - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\sin\nu\phi d\phi\right] = 0$$

we conclude $A_{n\nu} = 0$. Similarly, noting

$$\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\nu\phi d\phi - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos\nu\phi d\phi\right] = \frac{4(-1)^m}{2m+1}, \qquad m = 0,1,2,\ldots.$$

where I've recognized that ν must be odd, ie, $\nu = 2m+1$. Also

$$\int_0^L \sin(\frac{n\pi z}{L})dz = \frac{2}{(2l+1)\pi}, \qquad l = 0,1,2,\ldots..$$

where again I've recognized that n must be odd, ie, $n = 2l+1$. Thus

$$B_{n\nu} = \frac{16(-1)^m V}{\pi^2 I_{2m+1}(\frac{n\pi b}{L})(2l+1)(2m+1)}$$

b) Now $z = L/2, \quad L >> b,\ L >> \rho.$ Then from the class notes

$$I_{2m+1}\left(\frac{(2l+1)\pi\rho}{L}\right) \sim \frac{1}{\Gamma(2m+2)}\left[\frac{(2l+1)\pi\rho}{2L}\right]^{m+1}$$

Also

$$\sin\left[\frac{(2l+1)\pi}{L}\right] = (-1)^{l}$$

so

$$\Phi(\rho,z,\phi) = \sum_{l,m} \frac{16(-1)^{l+m}V}{\pi^2(2l+1)(2m+1)}\left(\frac{\rho}{b}\right)^{2m+1}\cos[(2m+1)\phi]$$

Using

$$\tan^{-1(x)} = \sum_{l=0}^{\infty} \frac{x^{2l+1}}{1l+1}(-1)^{l}$$

$$\frac{\pi}{4} = \tan^{-1}(1) = \sum_{l=0}^{\infty} \frac{(-1)^{l}}{2l+1}$$

so

$$\Phi(\rho,z,\phi) = \frac{4V}{\pi}\sum_{m} \frac{(-1)^{m}}{2m+1}\left(\frac{\rho}{b}\right)^{2m+1}\cos[(2m+1)\phi]$$

Remembering from problem 2.13 that

$$\sum_{m} \frac{(-1)^{m}}{2m+1}\left(\frac{\rho}{b}\right)^{2m+1}\cos[(2m+1)\phi] = \frac{1}{2}\tan^{-1}\left[\frac{2\left(\frac{\rho}{b}\right)\cos\phi}{\left(1-\frac{\rho^2}{b^2}\right)}\right]$$

we find

$$\Phi(\rho,z,\phi) = \frac{2V}{\pi}\tan^{-1}\left[\frac{2\left(\frac{\rho}{b}\right)\cos\phi}{\left(1-\frac{\rho^2}{b^2}\right)}\right]$$

which is the answer for problem 2.13.