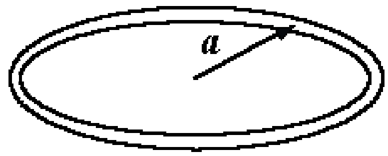
PHY 5346
HW Set 6 Solutions – Kimel

1. 3.12 The system is described by



a) From Eq. (3.106)

$$\Phi(\rho,\phi,z)=\sum_{m=0}^{\infty}\int_0^{\infty}dke^{-kz}J_m(k\rho)\left[A_m(k)\sin m\phi+B_m(k)\cos m\phi\right]$$

where from Eq. (3.109),

$$\left.\begin{matrix}A_m(k)\\B_m(k)\end{matrix}\right\}=\frac{k}{\pi}\int_0^{\infty}d\rho\rho\int_0^{2\pi}d\phi V\left(\rho,\phi\right)J_m(k\rho)\left\{\begin{matrix}\sin m\phi\\ \cos m\phi\end{matrix}\right.$$

where we use $\frac{1}{2}B_0$ for $m=0$.

b) Using cylidrical coordinates, with the origin at the center of the disc, then we have $\rho=0$, and can use the small argument expansion for $J_m(k\rho)$

$$J_m(k\rho)|_{\rho=0}=\frac{\delta_{m0}}{\Gamma(1)}+O((k\rho)^2)=\delta_{m0}$$

$$\Phi(0,\phi,z)=\frac{1}{2}\int_0^{\infty}dke^{-kz}B_0(k)$$

And, using Mathematica 4,

$$B_0(k)=2kV\int_0^a d\rho\rho J_0(k\rho)=2kV\frac{a}{k}J_1(ka)=2VaJ_1(ka)$$

Thus, again using Mathematica 4,

$$\Phi(0,\phi,z)=Va\int_0^{\infty}dke^{-kz}J_1(ka)=V\frac{\sqrt{z^2+a^2}-z}{\sqrt{z^2+a^2}}=V\left(1-\frac{z}{\sqrt{z^2+a^2}}\right)$$

c) We notice that for this $V(\rho,\phi)$, which is independent of ϕ, that all $A_m(k)$ vanish, and that only B_0 is nonzero. Again

$$B_0(k)=2kV\int_0^a d\rho\rho J_0(k\rho)=2kV\frac{a}{k}J_1(ka)=2VaJ_1(ka)$$

$$\Phi(a,\phi,z)=Va\int_0^{\infty}dke^{-kz}J_0(ka)J_1(ka)$$

Using Mathematic 4,

$$\int_0^\infty dke^{-kz} J_0(ka) J_1(ka) = \frac{1}{2a}\left(1 - \frac{zk}{\pi a} K(k)\right)$$

where $k = \frac{2a}{\sqrt{z^2+a^2}}$, and the complete elliptic integral of the first kind is defined by

$$K(k) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}$$

Thus

$$\Phi(a,\phi,z) = \frac{V}{2}\left(1 - \frac{zk}{\pi a} K(k)\right)$$