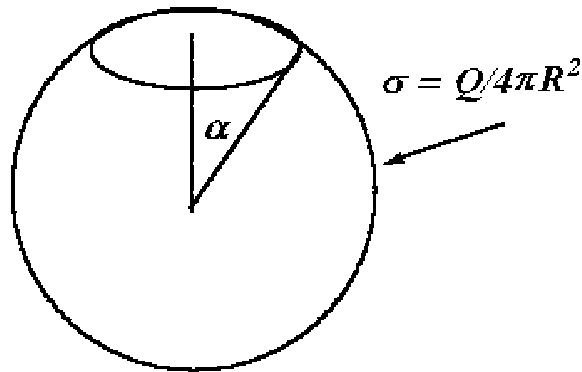


1. 3.2 The charge distribution is shown by



a) We see the charge distribution is given by

$$\rho(\vec{r}) = N\theta(\cos\alpha - \cos\theta)\delta(r - a)$$

where N is determined by the requirement $\int d^3r\rho(\vec{r}) = Q$, or

$$\rho(\vec{r}) = \frac{Q}{4\pi R^2}\theta(\cos\alpha - \cos\theta)\delta(r - R)$$

Expanding $\theta(\cos\alpha - \cos\theta)$ in terms of Legendre polynomials,

$$\theta(\cos\alpha - \cos\theta) = \sum_l A_l P_l(\cos\theta)$$

or

$$A_l = \frac{2l+1}{2} \int_{-1}^{\cos\alpha} P_l(x) dx$$

Using Mathematica 4, I get

$$\int_{-1}^{\cos\alpha} P_l(x) dx = \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1}$$

Notice for $l = 0$ in the above, $P_{-1}(\cos\alpha) \equiv -1$.

$$A_l = \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2}$$

This problem has azimuthal symmetry, so we can write in general (when $r < R$)

$$\phi(\vec{r}) = \sum_l B_l r^l P_l(\cos\theta)$$

where now θ is the polar angle of the vector \vec{r} . Choosing $\vec{r} \parallel \hat{z}$,

$$\phi(\vec{r} = r\hat{z}) = \sum_l B_l r^l P_l(1)$$

On the other hand we can write

$$\begin{aligned} \phi(\vec{r} = r\hat{z}) &= \frac{1}{4\pi\epsilon_0} \frac{Q}{4\pi R^2} \sum_l \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2} \int \frac{d\phi d\cos\theta r'^2 dr' P_l(\cos\theta) \delta(r' - R)}{|\vec{r}' - \vec{r}|} \\ &= \frac{1}{4\pi\epsilon_0} \frac{Q}{4\pi R^2} 2\pi \sum_{l'} \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2} \frac{r^l R^2}{R^{l+1}} \int_{-1}^1 dx P_l(x) P_{l'}(x) \end{aligned}$$

Using $\int_{-1}^1 dx P_l(x) P_{l'}(x) = \frac{2}{2l+1} \delta_{ll'}$

$$\phi(\vec{r} = r\hat{z}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2} \sum_l \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1} \frac{r^l}{R^{l+1}}$$

Then for general directions of \vec{r} ,

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2} \sum_l \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1} \frac{r^l}{R^{l+1}} P_l(\cos\theta)$$

If \vec{r} is on the outside, we know that R and r are interchanged in the expansion

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2} \sum_l \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1} \frac{R^l}{r^{l+1}} P_l(\cos\theta)$$

b) By symmetry, at the origin the electric field is along \hat{z} .

$$\begin{aligned} \vec{E}(0) &= -\frac{\partial}{\partial z} \phi(0) \hat{z} = \\ &= -\hat{z} \frac{\partial}{\partial z} \frac{1}{4\pi\epsilon_0} \frac{Q}{2} \frac{(P_2(\cos\alpha) - P_0(\cos\alpha))}{3R^2} z + \text{terms that vanish} \\ \vec{E}(0) &= \frac{Q}{12\pi\epsilon_0 R^2} (1 - P_2(\cos\alpha)) \end{aligned}$$

c) Consider the case where α is very small. Using our general expression for $\phi(\vec{r})$, we see we need to expand $P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)$. (I will keep the leading terms.)

$$P_l(\cos\alpha) = \sum_n \frac{1}{n!} \frac{d^n}{dx^n} P_l(x)|_{x=1} (x-1)^n \approx P_l(1) + \frac{d}{dx} P_l(x)|_{x=1} (x-1)$$

$$((x-1) = \sqrt{1-\epsilon^2} - 1) = -\frac{1}{2}\epsilon^2 + O(\epsilon^4)$$

where $\varepsilon = \sin \alpha$.

$$P_l(\cos \alpha) = 1 - \frac{1}{2} \frac{d}{dx} P_l(1) \varepsilon^2$$

$$P_{l+1}(\cos \alpha) - P_{l-1}(\cos \alpha) = \frac{\varepsilon^2}{2} \left(\frac{d}{dx} P_{l-1}(1) - \frac{d}{dx} P_{l+1}(1) \right) = -\frac{\varepsilon^2 (2l+1) P_l(1)}{2}$$

where I have used Eq. (3.28) and these formulas apply for $l > 0$.

So

$$\phi(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 R} - \frac{1}{4\pi\varepsilon_0} \frac{Q\varepsilon^2}{4} \sum_l \frac{r^l}{R^{l+1}} P_l(\cos \theta) = \frac{Q}{4\pi\varepsilon_0 R} - \frac{1}{4\pi\varepsilon_0} \frac{Q\varepsilon^2}{4|R\hat{z} - \vec{r}'|}$$

That is, the potential is just that of a uniformly charged sphere plus a point charge $= -Q \frac{(\text{solid angle subtended by empty cap})}{4\pi}$, located at the point $R\hat{z}$.

The electric field for this point charge is obviously given by

$$\vec{E}(\vec{r}) = \frac{-1}{4\pi\varepsilon_0} \frac{Q\varepsilon^2 (\vec{r}' - R\hat{z})}{4|R\hat{z} - \vec{r}'|^3}$$

If the charge were located on a small cap at the bottom of the sphere, ie, if $\alpha \rightarrow \pi - \beta$, then clearly in analogy with what we have already done, we can see that it would act like a point charge $= Q \frac{(\text{solid angle subtended by cap})}{4\pi}$

and located at the point $-R\hat{z}$. Then the potential is

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{Q\varepsilon^2}{4|R\hat{z} + \vec{r}'|}$$

where now $\varepsilon = \sin \beta$.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{Q\varepsilon^2 (\vec{r}' + R\hat{z})}{4|R\hat{z} + \vec{r}'|^3}$$