PHY 5346
HW Set 5 Solutions – Kimel

4. 3.4 Slice the sphere equally by n planes slicing through the z axis, subtending angle $\Delta\phi$ about this axis with the surface of each slice of the pie alternating as $\pm V$.

$$\phi(r,\theta,\phi) = \sum_{l,m} A_{lm} r^l Y_l^m(\theta,\phi)$$

so

$$A_{lm} = \frac{1}{a^l}\int d\Omega (Y_l^m(\theta,\phi))^* \phi(a,\theta,\phi)$$

Symmetries:

$$A_{l-m} = (-1)^m (A_{lm})^*$$

$$\phi(r,\theta,\phi+2\Delta\phi) = \phi(r,\theta,\phi)$$

where

$$\Delta\phi = \frac{2\pi}{2n}$$

Thus

$$m = \pm n, \text{ and integral multiples thereof}$$

$$\phi(-\vec{r}) = -\phi(\vec{r}),\ n = 1$$

$$\phi(-\vec{r}) = \phi(\vec{r}),\ n > 1$$

Since

$$PY_l^m(\theta,\phi) = (-1)^l Y_l^m(\theta,\phi)$$

Then

$$l \text{ is odd for } n = 1;\ \ l \text{ is even for } n > 1$$

Thus we only have contributions of $l \geq n$. Using

$$A_{lm} = \frac{1}{a^l}\int d\Omega (Y_l^m(\theta,\phi))^* \phi(a,\theta,\phi)$$

The integral over ϕ can be done trivially, since the integrand is just $e^{-im\phi}$ leaving the desired answer in terms of an integral over $\cos\theta$.

$n = 1$ case: I am going to keep only the lowest novanishing terms, involving A_{11} and A_{1-1}.

$$\phi = r(A_{11}Y_1^1 + A_{1-1}Y_1^{-1}) = r(A_{11}Y_1^1 + (A_{11}Y_1^1)^*) = 2r\,\mathrm{Re}(A_{11}Y_1^1)$$

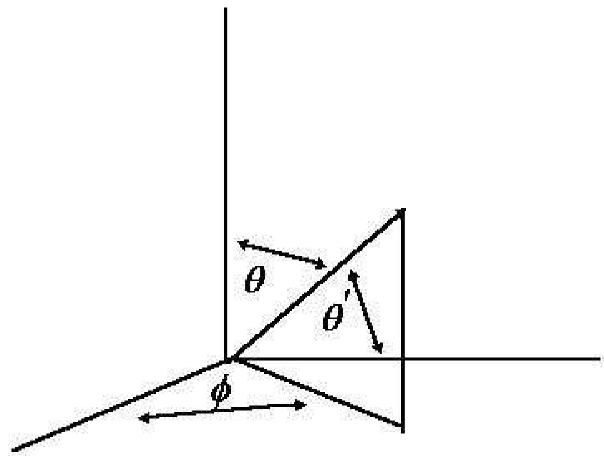
$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}(1-x^2)^{1/2}e^{i\phi}$$

$$A_{11} = -\frac{1}{a}\sqrt{\frac{3}{8\pi}}V\left[\int_{-1}^{1}(1-x^2)^{1/2}dx\right]\left[\int_0^{\pi} e^{-i\phi}d\phi - \int_{\pi}^{2\pi} e^{-i\phi}d\phi\right]$$

$$A_{11} = \frac{2i\pi}{a}\sqrt{\frac{3}{8\pi}}V$$

$$\phi = 2r\,\mathrm{Re}\left[\left(\frac{2i\pi}{a}\sqrt{\frac{3}{8\pi}}V\right)\left(-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}\right)\right] = \frac{3r}{2a}V\sin\theta\sin\phi$$

From the figure



we see

$$\sin\theta\sin\phi = \cos\theta'$$

So

$$\phi = \frac{3r}{2a}V\cos\theta' = V\left[\frac{3}{2}\frac{r}{a}P_1(\cos\theta') + \ldots\ldots\right]$$

The other terms, for $l = 2, 3,$ can be obtained in the same way in agreement with the result of (3.36)