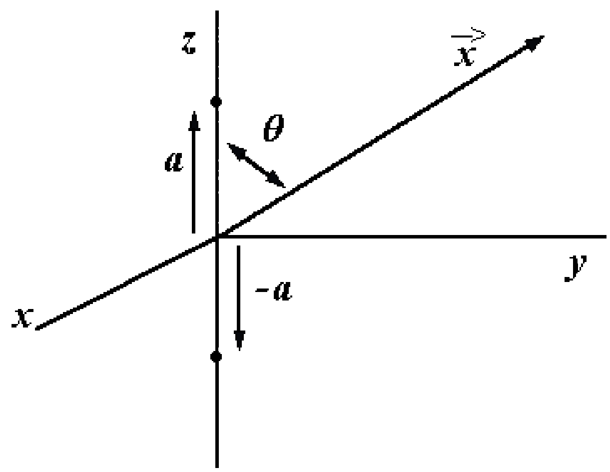
PHY 5346
HW Set 5 Solutions – Kimel

3. 3.6 The system is described by



a) From the figure, we can write

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{x}-\vec{a}|} - \frac{1}{|\vec{x}+\vec{a}|}\right]$$

And using the familiar expansion of $\frac{1}{|\vec{x}-\vec{a}|}$, this expression can be written

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\sum_l \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l [P_l(\theta) - P_l(\pi-\theta)]$$

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\sum_l \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l \left(1+(-1)^{l+1}\right) P_l(\theta)$$

This can be written in terms of spherical harmonics using $P_l(\theta) = \sqrt{\frac{4\pi}{2l+1}}Y_l^0(\theta,\phi)$.
b) We are given $r > a$, so

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\sum_l \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l \left(1+(-1)^{l+1}\right) P_l(\theta)$$

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\sum_l \frac{1}{r}\left(\frac{a}{r}\right)^l \left(1+(-1)^{l+1}\right) P_l(\theta); a \to 0 = \frac{q2a}{4\pi\varepsilon_0 r^2}P_1(\theta)$$

$$\phi(\vec{x}) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

c) The electric dipole is the particular solution, and ϕ_0 is the homogeneous solution which is a solution to Laplace's equation: by superposition,

$$\phi(\vec{x}) = \phi_p + \phi_0$$

$$\phi(\vec{x}) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} + \sum_l A_l r^l P_l(\theta)$$

The boundary condition we must satisfy is that $\phi(|\vec{x}| = b) = 0$, so

$$\frac{p\cos\theta}{4\pi\varepsilon_0 b^2} + \sum_l A_l b^l P_l(\theta) = 0$$

$$\rightarrow A_l = \begin{matrix} 0, \ l \neq 1 \\ -\frac{p}{4\pi\varepsilon_0 b^3}, \ l = 1 \end{matrix}$$

$$\phi(\vec{x}) = \frac{p\cos\theta}{4\pi\varepsilon_0}\left(\frac{1}{r^2} - \frac{r}{b^3}\right)$$