PHY 5346
HW Set 7 Soutions – Kimel

2. 4.6

a) We know that

$$W = -\frac{1}{6}\sum_i Q_{ii}\frac{\partial}{\partial x_i}E_i(0)$$

The problem is cylindrically symmetric, so $Q_{11} = Q_{22}$. Using the fact that the trace of the quadrupole tensor is zero, we see

$$Q_{11} = Q_{22} = -\frac{1}{2}Q_{33}$$

The book defines the quadrupole moment in nucleii to be $Q = \frac{1}{e}Q_{33}$. The electric field in our formula for W refers to the external electric field, so within the nucleus $\vec{\nabla}\cdot\vec{E} = 0$, or

$$\frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y = -\frac{\partial}{\partial z}E_z$$

Thus

$$W = -\frac{eQ}{6}\left(\frac{\partial}{\partial z}E_z - \frac{1}{2}\left(\frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y\right)\right)_0 = -\frac{eQ}{6}\left(\frac{\partial}{\partial z}E_z - \frac{1}{2}\left(-\frac{\partial}{\partial z}E_z\right)\right)_0$$

$$W = -\frac{eQ}{6}\left(\frac{\partial}{\partial z}E_z\right)_0\left(1+\frac{1}{2}\right) = -\frac{eQ}{4}\left(\frac{\partial}{\partial z}E_z\right)_0$$

b)

$$\left(\frac{\partial}{\partial z}E_z\right)_0 = -\frac{4W}{eQ} = -\frac{4W}{eQ\left(\frac{e}{4\pi\varepsilon_0 a_0^3}\right)}\left(\frac{e}{4\pi\varepsilon_0 a_0^3}\right)$$

Now from the particle data book,

$$\frac{e^2}{4\pi\varepsilon_0} = \alpha\hbar c = \frac{\alpha h c}{2\pi}, \text{ with } \alpha = 1/137$$

So

$$\frac{4W}{eQ\left(\frac{e}{4\pi\varepsilon_0 a_0^3}\right)} = \frac{4\,(W/h)\,2\pi a_0^3}{Q\alpha c} = \frac{4\cdot 10^7\text{sec}^{-1}2\pi\left(0.529\times 10^{-10}\right)^3\text{m}^3}{2\times 10^{-28}\text{m}^2\,(1/137)\times 3\times 10^8\text{m/sec}} = 0.085$$

$$\left(\frac{\partial}{\partial z}E_z\right)_0 = -0.085\left(\frac{e}{4\pi\varepsilon_0 a_0^3}\right)$$

c) Let us assume the spheroid is gotten by a rotation about the semimajor axis. The equation for a spheroid is given by

$$\frac{x^2+y^2}{b^2} + \frac{z^2}{a^2} = 1$$

The volume of the spheroid is

$$V = \int_0^{2\pi} d\phi \int_0^b \rho d\rho \int_{-a\sqrt{1-\rho^2/b^2}}^{a\sqrt{1-\rho^2/b^2}} dz = \frac{4\pi}{3} ab^2$$

where $\rho^2 = x^2 + y^2$.
Thus the charge density of the nucleus is

$$\rho_c = \frac{3Ze}{4\pi ab^2}$$

$$Q_{33} = \rho_c 2\pi \int_0^b \rho d\rho \int_{-a\sqrt{1-\rho^2/b^2}}^{a\sqrt{1-\rho^2/b^2}} \left(2z^2 - \rho^2\right) dz$$

$$Q_{33} = \rho_c 2\pi \int_0^b \rho \left(\frac{2}{3} a \sqrt{\left(\frac{b^2 - \rho^2}{b^2}\right)} \frac{2a^2b^2 - 2a^2\rho^2 - 3\rho^2 b^2}{b^2}\right) d\rho = \rho_c 2\pi \frac{4ab^2\left(a^2 - b^2\right)}{15}$$

$$Q_{33} = \left(\frac{3Ze}{4\pi ab^2}\right) 2\pi \frac{4ab^2\left(a^2 - b^2\right)}{15} = \frac{2}{5} Ze\left(a^2 - b^2\right)$$

So

$$Q = \frac{2}{5} Z\left(a^2 - b^2\right) = \frac{4}{5} Z\left(a - b\right)\left(a + b\right)/2 = \frac{4}{5} ZR\left(a - b\right)$$

Or

$$\frac{(a-b)}{R} = \frac{5Q}{4ZR^2} = \frac{5 \cdot 2.5 \times 10^{-28} \mathrm{m}^2}{4 \cdot 63 \cdot \left(7 \times 10^{-15}\right)^2 \mathrm{m}^2} = 0.101$$