

3. 4.7 a) Since ρ does not depend on ϕ , we can write it in terms of spherical harmonics with $m = 0$. First note

$$Y_2^0 = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} (1 - \sin^2\theta) - \frac{1}{2} \right)$$

or

$$\sin^2\theta = -\frac{2}{3} \sqrt{\frac{4\pi}{5}} Y_2^0 + \sqrt{4\pi} \frac{2}{3} Y_0^0$$

Thus only the $m = 0, l = 0, 2$ multipoles contribute.

$$q_{00} = \frac{2\sqrt{4\pi}}{3} \int_0^\infty r^2 \left(\frac{1}{64\pi} r^2 e^{-r} \right) dr = \frac{2\sqrt{4\pi}}{3} \frac{3}{8\pi} = \frac{1}{2\sqrt{\pi}}$$

$$q_{20} = -\frac{2}{3} \sqrt{\frac{4\pi}{5}} \int_0^\infty r^4 \left(\frac{1}{64\pi} r^2 e^{-r} \right) dr = -\frac{2}{3} \sqrt{\frac{4\pi}{5}} \frac{45}{4\pi} = -3 \frac{\sqrt{5}}{\sqrt{\pi}}$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \left[4\pi q_{00} \frac{Y_0^0}{r} + 4\pi q_{20} \frac{Y_2^0}{5r^3} \right] = \frac{1}{4\pi\epsilon_0} \left[\sqrt{4\pi} q_{00} \frac{P_0}{r} + \sqrt{\frac{4\pi}{5}} q_{20} \frac{P_2}{r^3} \right]$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \left[\frac{P_0}{r} - 6 \frac{P_2}{r^3} \right]$$

b)

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}') d^3x'}{|\vec{x} - \vec{x}'|}$$

Using

$$\frac{1}{|\vec{x} - \vec{x}'|} = 4\pi \sum_{lm} \frac{1}{(2l+1)r_>} \left(\frac{r_<}{r_>} \right)^l Y_l^{m*}(\theta', \phi') Y_l^m(\theta, \phi)$$

, we see only the $l = 0, 2$ and $m = 0$ terms of the expansion contribute in the potential. Next take $r' > r$.

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} 4\pi \sum_{lm} \frac{1}{(2l+1)} r^l Y_l^m(\theta, \phi) \int Y_l^{m*}(\theta', \phi') r'^2 d\Omega' \frac{\rho(\vec{x}')}{r'^{l+1}} dr'$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} 4\pi \left[Y_0^0 \sqrt{4\pi} \frac{2}{3} \int_0^\infty \left(\frac{1}{64\pi} r^2 e^{-r} \right) r dr + \frac{Y_2^0}{5} r^2 \left(-\frac{2}{3} \sqrt{\frac{4\pi}{5}} \int_0^\infty \left(\frac{1}{64\pi} r^2 e^{-r} \right) \frac{1}{r} dr \right) \right]$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} 4\pi \left[Y_0^0 \sqrt{4\pi} \frac{2}{3} \frac{3}{32\pi} + \frac{Y_2^0}{5} r^2 \left(-\frac{2}{3} \sqrt{\frac{4\pi}{5}} \right) \frac{1}{64\pi} \right]$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} 4\pi \left[P_0 \frac{2}{3} \frac{3}{32\pi} + \frac{P_2}{5} r^2 \left(-\frac{2}{3} \right) \frac{1}{64\pi} \right] = \frac{1}{4\pi\epsilon_0} \left[\frac{P_0}{4} - \frac{r^2 P_2}{120} \right]$$