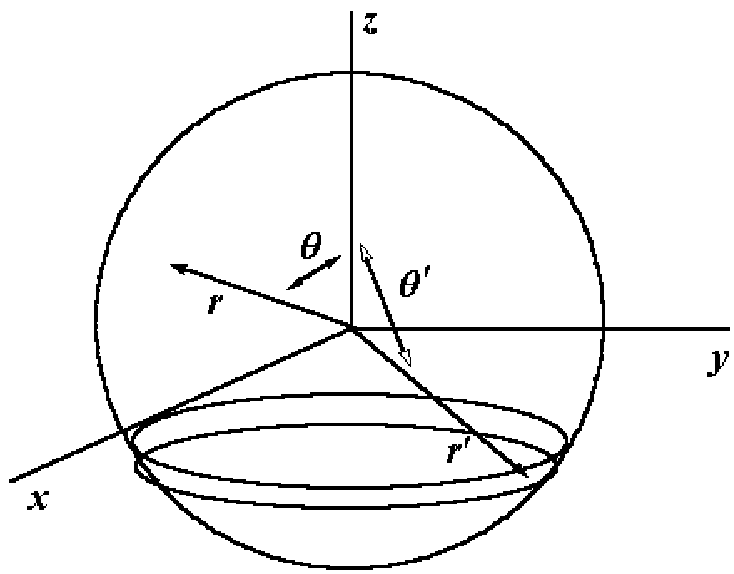
PHY 5346
HW Set 8 Solutions – Kimel

3. 5.13 We may choose the coordinate system so the $\vec{r}$ lies in the $x-z$ plane:



The vector potential is given by

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}d^3x'}{|\vec{r}-\vec{r}'|}$$

Noting $\vec{J}d^3x' \to \Delta I d\vec{l}'$, where

$$\Delta I = \frac{\Delta Q}{\tau} = \frac{\sigma a^2 d\Omega'}{2\pi/\omega}$$

$$d\vec{l}' = a\left|\sin\theta'\right| d\phi'\hat{\phi}'$$

Since

$$\hat{\phi}' = \cos\phi'\hat{y} - \sin\phi'\hat{x}$$

By symmetry, the x-component of $\vec{A}$ vanishes, so

$$A_y = \frac{\mu_0}{4\pi}\sigma\omega a^3 \int \frac{\sin\theta' \cos\phi' d\Omega'}{|\vec{r}-\vec{r}'|}$$

where I've used

$$Y_1^1(\theta',\phi') = -\sqrt{\frac{3}{8\pi}}\sin\theta' e^{im\phi'}$$

$$A_y = \frac{\mu_0}{4\pi}\sigma\omega a^3 \left(-\frac{1}{2}\sqrt{\frac{8\pi}{3}}\right) \int \frac{(Y_1^1(\theta', \phi') + Y_1^{*1}(\theta', \phi'))d\Omega'}{|\vec{r} - \vec{r'}|}$$

Using the expansion

$$\frac{1}{|\vec{r} - \vec{r'}|} = \sum_{l,m} \frac{4\pi}{2l+1} Y_l^{m*}(\theta', \phi') Y_l^m(\theta, 0)$$

and the fact that $Y_l^m(\theta, 0)$ is real, we see only the $l = 1$, $m = 1$ terms contribute.

$$A_y = \frac{\mu_0}{4\pi}\sigma\omega a^3 \left(-\sqrt{\frac{8\pi}{3}}\right) \frac{4\pi}{3} \frac{r_<}{r_>^2} Y_1^1(\theta, 0) = \frac{\mu_0}{4\pi}\sigma\omega a^3 \frac{4\pi}{3} \frac{r_<}{r_>^2} \sin\theta$$

If we take $\vec{r}$ to be in an arbitrary direction $A_y \to A_\phi$. Also, noting $Q = 4\pi a^2$,

$$A_\phi = \frac{\mu_0}{4\pi} \frac{Q\omega a}{3} \frac{r_<}{r_>^2} \sin\theta$$

Thus on the
inside:

$$A_\phi = \frac{\mu_0}{4\pi} \frac{Q\omega}{3} \frac{r}{a} \sin\theta$$

outside:

$$A_\phi = \frac{\mu_0}{4\pi} \frac{Q\omega}{3} \frac{a^2}{r^2} \sin\theta$$

Remembering for this case

$$\vec{B} = \vec{\nabla} \times \vec{A} = \hat{r}\left[\frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta A_\phi)\right] + \hat{\theta}\left[-\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]$$

Thus on the

inside:

$$B_r = \frac{\mu_0}{4\pi} \frac{Q\omega}{3} \frac{2\cos\theta}{a}, \qquad B_\theta = -\frac{\mu_0}{4\pi} \frac{Q\omega}{3} \frac{2\sin\theta}{a}$$

outside:

$$B_r = \frac{\mu_0}{4\pi} \frac{Q\omega}{3} \frac{a^2 2\cos\theta}{r^3}, \qquad B_\theta = \frac{\mu_0}{4\pi} \frac{Q\omega}{3} \frac{a^2\sin\theta}{r^3}$$