PHY 5346
Homework Set 10 Solutions – Kimel

4. 5.14 This problem corresponds to $\vec{J} = 0$, so we have the equations

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \text{and } \vec{\nabla} \times \vec{H} = 0$$

from which it follow that

$$\vec{H} = -\vec{\nabla}\Phi$$

where $\vec{B} = \mu\vec{H}$. From the first two equations we have the boundary conditions at an interface:

$$B_{1\perp} = B_{2\perp}, \quad \text{and } H_{1||} = H_{2||}$$

From the discussion on p. 76 of the text, the potential is independent of $z$ and can be expanded as

$$\Phi(\rho, \phi) = \sum_m \left[A_m \rho^m + B_m \rho^{-m}\right] (C_m \sin m\phi + D_m \cos m\phi)$$

Because the system is odd under reflection through the $y$ axis, which I take to be along $\vec{B}_0$, there are no cosine terms in the expansion. In the region III, outside the cylinder, as $\rho \to \infty$, $-\vec{\nabla}\Phi = \vec{H} = H_0\hat{y}$. Thus $\Phi_{III} \to -H_0 y = -H_0 \rho \sin\phi$. Here $H_0 = B_0/\mu_0$. The boundary conditions can be satisfied if only the $m = 1$ terms are kept in the expansion, and we know that the solution which satisfies the boundary conditions is unique. Thus we have the expansions

Region I, $\rho < a$ :

$$\Phi_I = A\rho \sin\phi$$

Region II, $a < \rho < b$.

$$\Phi_{II} = \left[C\rho + D\rho^{-1}\right] \sin\phi$$

Region III, $b < \rho$.

$$\Phi_{III} = -H_0 \rho \sin\phi + E\rho^{-1} \sin\phi$$

Applying the boundary conditions, we have the four conditions

$$\Phi_I|_{\rho=a} = \Phi_{II}|_{\rho=a}$$

$$\mu_0 \frac{\partial}{\partial\rho}\Phi_I|_{\rho=a} = \mu \frac{\partial}{\partial\rho}\Phi_{II}|_{\rho=a}$$

$$\Phi_{II}|_{\rho=b} = \Phi_{III}|_{\rho=b}$$

$$\mu \frac{\partial}{\partial\rho}\Phi_{II}|_{\rho=b} = \mu_0 \frac{\partial}{\partial\rho}\Phi_{III}|_{\rho=b}$$

These four boundary conditions allow us to solve for $A, C, D$, and $E$, with the result that

$$A = \frac{4H_0 b^2 \mu_r}{d}$$

$$C = \frac{2H_0 b^2 (\mu_r + 1)}{d}$$

$$D = \frac{2H_0 (\mu_r - 1) a^2 b^2}{d}$$

$$E = \frac{H_0 b^2 \left[2(\mu_r + 1) b^2 + 2(\mu_r - 1) a^2 + d\right]}{d}$$

where

$$d = a^2 (\mu_r - 1)^2 - b^2 (\mu_r + 1)^2$$

and the relative permealbility is

$$\mu_r = \frac{\mu}{\mu_0}$$

With these expressions

$$\bar{B}_I = -\mu_0 \vec{\nabla} \Phi_I$$

$$\bar{B}_{II} = -\mu \vec{\nabla} \Phi_{II}$$

$$\bar{B}_{III} = -\mu_0 \vec{\nabla} \Phi_{III}$$