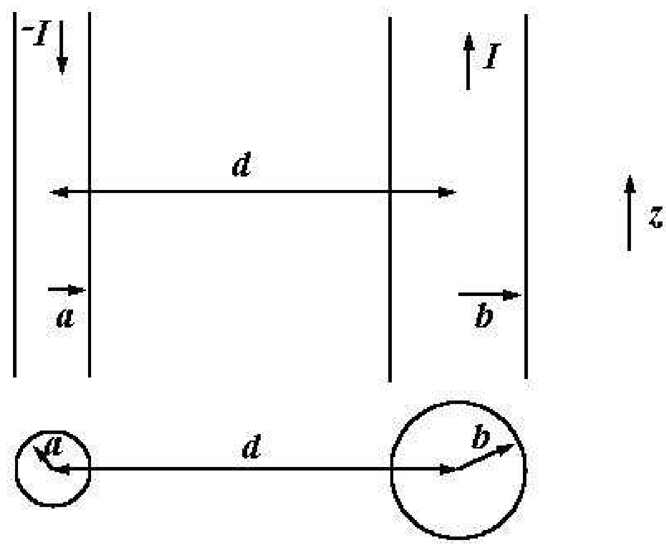
PHY 5346
Homework Set 11 Solutions – Kimel

1. 5.26 The system is described by



Since the wires are nonpermeable, $\mu = \mu_0$. The system is made of parts with cylindrical symmetry, so we can determine B using Ampere's law.

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}, \text{ or } \quad \int \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{a}$$

On the **outside** of each wire,

$$\int \vec{B} \cdot d\vec{l} = B 2\pi\rho = \mu_0 I \to B_{out} = \frac{\mu_0 I}{2\pi\rho}$$

On the **inside** of each wire

$$\int \vec{B} \cdot d\vec{l} = B 2\pi\rho = \mu_0 I \frac{\rho^2}{R^2}, \quad B_{in} = \frac{\mu_0 I}{2\pi} \frac{\rho}{R^2} \text{ with } R = a, b$$

From the right-hand rule, the B from each wire is in the $\hat{\phi}$ direction. From the above figure, using the general expression for the vector potential, we see $\vec{A}$ is in the $\pm\hat{z}$ direction. Since $\vec{\nabla} \times \vec{A} = \vec{B}$,

$$B_z = -\frac{\partial}{\partial\rho} A_z \to A_z = -\int B_z d\rho$$

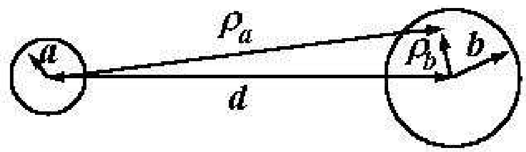
Thus

$$A_z = \begin{cases} -\frac{\mu_0 I}{2\pi}\left(\ln\frac{\rho}{R} + C\right) = -\frac{\mu_0 I}{4\pi}\left(\ln\frac{\rho^2}{R^2} + 1\right) & \text{on the outside} \\ -\frac{\mu_0 I}{4\pi}\frac{\rho^2}{R^2}, & \text{on the inside} \end{cases}$$

where I've determined $C = 1/2$, from the requirement that A_z be continuous at $\rho = R$. Let l be the length of the wire. Then we know the total potential energy is given by

$$W = \frac{1}{2}\int \vec{J}\cdot\vec{A}d^3x = \frac{l}{2}\int[J_a A da_a + J_b A da_b]$$

Consider the second term $\frac{l}{2}\int J_b A da_b$. The system is pictured as



From the figure

$$\vec{\rho}_a = \vec{d} + \vec{\rho}_b, \quad \rho_a^2 = d^2 + \rho_b^2 - 2d\rho_b\cos\phi$$

so, since $J_b = \frac{I}{\pi b^2}$

$$\frac{l}{2}\int J_b A da_b = \frac{l}{2}\frac{I}{\pi b^2}\int[A_{out}(\rho_a) + A_{in}(\rho_{b)})]\rho_b d\rho_b d\phi$$

$$= \frac{l}{2}\frac{I}{\pi b^2}\frac{\mu_0 I}{4\pi}\int\left[\ln\frac{\rho_a^2}{a^2} + 1 - \frac{\rho_b^2}{b^2}\right]\rho_b d\rho_b d\phi$$

$$\simeq \frac{l}{2}\frac{I}{\pi b^2}\frac{\mu_0 I}{4\pi}2\pi\int_0^b\left(\ln\frac{d^2}{a^2} + 1 - \frac{\rho_b^2}{b^2}\right)\rho_b d\rho_b$$

$$= \frac{l}{2}\frac{I}{\pi b^2}\frac{\mu_0 I}{4\pi}2\pi\frac{1}{4}b^2\left(1 + 2\ln\frac{d^2}{a^2}\right) = \frac{l}{2}\left(\frac{\mu_0}{4\pi}\right)\left(\frac{1}{2} + 2\ln\frac{d}{a}\right)I^2$$

The first term $\frac{l}{2}\int J_a A da_a$ is equal to

$$\frac{l}{2}\int J_a A da_a = \frac{l}{2}\left(\frac{\mu_0}{4\pi}\right)\left(\frac{1}{2} + 2\ln\frac{d}{b}\right)I^2$$

Thus

$$W = \frac{l}{2}\left(\frac{\mu_0}{4\pi}\right)\left(1 + 2\ln\frac{d^2}{ab}\right)I^2 = \frac{l}{2}\frac{L}{l}I^2$$

or

$$\frac{L}{l} = \frac{\mu_0}{4\pi}\left(1 + 2\ln\frac{d^2}{ab}\right)$$