PHY 5346
Homework Set 12 Solutions – Kimel

1. 6.21

a) I'm going to represent the dipole as a charge $-q$ at $\vec{r}_0$ and a charge q at $\vec{r}_0 + \vec{l}$. We take the limit

$$q\vec{l} \to \vec{p}$$

Thus

$$\rho = q\left[\delta(\vec{x} - \vec{r}_0 - \vec{l}) - \delta(\vec{x} - \vec{r}_0)\right]$$

Expanding around $\vec{l} = 0$ give

$$\rho(\vec{x}) = q\vec{\nabla}\delta(\vec{x} - \vec{r}_0) \cdot \left(-\vec{l}\right) = -\vec{p} \cdot \vec{\nabla}\delta(\vec{x} - \vec{r}_0)$$

As we've shown before for a collection of charges with charge density ρ and velocity $\vec{v}$

$$\vec{J} = \rho\vec{v} = -\vec{v}\left(\vec{p} \cdot \vec{\nabla}\right)\delta(\vec{x} - \vec{r}_0)$$

b) The magnetic dipole moment is given by

$$\vec{m} = \frac{1}{2}\int \vec{x} \times \vec{J} d^3x = -\frac{1}{2}\int \vec{x} \times \vec{v}\left(\vec{p} \cdot \vec{\nabla}\right)\delta(\vec{x} - \vec{r}_0)d^3x$$

Integrating by parts

$$\frac{1}{2}\vec{p} \cdot \int \vec{\nabla}\left(\vec{x} \times \vec{v}\right)\delta(\vec{x} - \vec{r}_0)d^3x$$

Look at the n^{th} component of the vector $\vec{p} \cdot \vec{\nabla}\left(\vec{x} \times \vec{v}\right)$

$$\left[\vec{p} \cdot \vec{\nabla}\left(\vec{x} \times \vec{v}\right)\right]_n = \sum_{ilm} p_i \partial_i \varepsilon_{lmn} x_l v_m = \sum_{lm} \varepsilon_{lmn} p_l v_m = \left[\vec{p} \times \vec{v}\right]_n$$

Thus

$$\vec{m} = \frac{1}{2}\int \vec{p} \times \vec{v}\left(\vec{x}\right)\delta(\vec{x} - \vec{r}_0)d^3x = \frac{1}{2}\vec{p} \times \vec{v}\left(\vec{r}_0\right)$$

Similarly

$$Q_{ij} = \int \left(3x_i x_j - r^2\delta_{ij}\right)\rho(\vec{x})d^3x = \int \left(3x_i x_j - r^2\delta_{ij}\right)\left[-\vec{p} \cdot \vec{\nabla}\delta(\vec{x} - \vec{r}_0)\right]d^3x$$

Integrating by parts

$$Q_{ij} = \sum_l \int p_l \partial_l \left(3x_i x_j - \sum_k x_k^2 \delta_{ij}\right)\delta(\vec{x} - \vec{r}_0)d^3x$$

$$Q_{ij} = \int \left(3p_i x_j + 3p_j x_i - 2\sum_l p_l x_l \delta_{ij}\right)\delta(\vec{x} - \vec{r}_0)d^3x$$

$$Q_{ij} = 3p_i x_{0j} + 3p_j x_{0i} - 2\vec{p} \cdot \vec{r}_0 \delta_{ij}$$