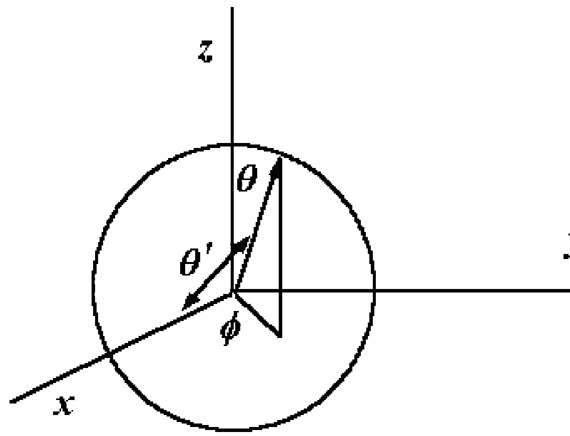
PHY 5346
Homework Set 12 Solutions – Kimel

1. 6.8

The physical system is shown as



We know from Maxwell's equations that $-\vec{\nabla}\cdot\vec{M}$ plays the role of the effective magnetic charge density. Using the fact that

$$\vec{M} = \frac{1}{2}\left(\vec{x}\times\vec{J}\right)$$

and the fact that $\vec{J} = \rho_{pol}\vec{v}$, where $\rho_{pol} = \delta(r-a)\,\sigma_{pol}$, where σ_{pol} is given by equation (4.58) of the textbook:

$$\sigma_{pol} = 3\varepsilon_0\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)E_0\cos\theta' = 3\varepsilon_0\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)E_0\sin\theta\cos\phi$$

Using the figure

$$\vec{M} = \frac{1}{2}\left(\vec{x}\times\vec{J}\right) = \frac{1}{2}\sigma_{pol}va\sin\theta\delta(r-a)\left(-\hat{\theta}\right)$$

Thus

$$\rho_m = -\vec{\nabla}\cdot\vec{M} = \sigma_{pol}v\cos\theta\delta(r-a) = a\omega 3\varepsilon_0\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)E_0\sin\theta\cos\phi\cos\theta\delta(r-a)$$

This can be written

$$\rho_m = a\omega 3\varepsilon_0\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)E_0\left(-\sqrt{\frac{8\pi}{15}}\right)\left(\frac{Y_2^1 - Y_2^{-1}}{2}\right)\delta(r-a)$$

Using

$$q_{lm} = \int Y_l^{m*}r^l\rho d^3x$$

there are only two moments which survive for this distribution

$$q_{2\pm1} = \pm a^5 \omega 3\varepsilon_0 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) E_0 \left(-\sqrt{\frac{8\pi}{15}}\right) \frac{1}{2}$$

Using (on the outside of the sphere)

$$\phi_m = \sum_{lm} \frac{1}{2l+1} q_{lm} \frac{Y_l^m}{r^{l+1}}$$

$$\phi_m = a^5 \omega 3\varepsilon_0 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) E_0 \left(-\sqrt{\frac{8\pi}{15}}\right) \frac{1}{5} \left(\frac{Y_2^1 - Y_2^{-1}}{2}\right)$$

Or

$$\phi_m = a^5 \omega 3\varepsilon_0 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) E_0 \left(-\sqrt{\frac{8\pi}{15}}\right) \frac{1}{5} \left(\frac{Y_2^1 - Y_2^{-1}}{2}\right) \frac{1}{r^3}$$

$$\phi_m = \frac{3}{5} \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) \omega \varepsilon_0 E_0 \left(\frac{a^5}{r^5}\right) xz$$

Repeat the same steps to get the potential on the inside of the sphere.