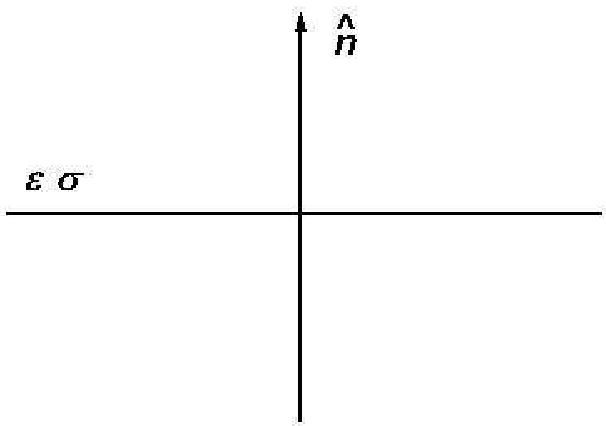
PHY 5346
Homework Set 13 Solutions – Kimel

2. 7.4 We have a nonpermeable conducting material, so $\mu = \mu_0$, and we have $J = \sigma E$, where $\sigma$ is the conductivity. The following figure describes the system:




The two boundary conditions that we must satisfy for plane waves are

$$E_0 + E_0'' - E_0' = 0$$

$$k(E_0 - E_0'') - k'E_0' = 0$$

Or

$$\frac{E_0''}{E_0} = \frac{k - k'}{k + k'}$$

We must take into account the fact that $\vec{J} = \sigma\vec{E}$. Adding in this term in Maxwell's equations for a plane wave, we get

$$k = \frac{\omega}{c}$$

$$k'^2 = \varepsilon\mu\omega^2\left(1 + i\frac{\sigma}{\omega\varepsilon}\right)$$

Thus we can write

$$k' = \sqrt{\varepsilon\mu}\,\omega(\alpha + i\beta)$$

with

$$\alpha = \left(\frac{\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1}{2}\right)^{1/2}$$

$$\beta = \left( \frac{\sqrt{1 + (\frac{\sigma}{\omega\varepsilon})^2} - 1}{2} \right)^{1/2}$$

Thus

$$\frac{E_0''}{E_0} = \frac{1 - \sqrt{\varepsilon\mu_0}\, c\alpha - i\sqrt{\varepsilon\mu_0}\, c\beta}{1 + \sqrt{\varepsilon\mu_0}\, c\alpha + i\sqrt{\varepsilon\mu_0}\, c\beta}$$

1) For a very poor conductor $\sigma$ is very small, so keeping only first order in $\sigma$

$$\alpha = \left( \frac{\sqrt{1 + (\frac{\sigma}{\omega\varepsilon})^2} + 1}{2} \right)^{1/2} \approx 1$$

$$\beta = \left( \frac{\sqrt{1 + (\frac{\sigma}{\omega\varepsilon})^2} - 1}{2} \right)^{1/2} \approx \frac{\sigma}{2\omega\varepsilon}$$

2) For the case of a very good conductor, $\frac{\sigma}{\omega\varepsilon} >> 1,$ so

$$\alpha \approx \sqrt{\frac{\sigma}{2\omega\varepsilon}} = \sqrt{\frac{\frac{2}{\mu_0\omega\delta^2}}{2\omega\varepsilon}} = \frac{1}{\omega\delta\sqrt{\mu_0\varepsilon}}$$

$$\beta \approx \sqrt{\frac{\sigma}{2\omega\varepsilon}} = \frac{1}{\omega\delta\sqrt{\mu_0\varepsilon}}$$

where I have used (5.165) to relate the conductivity to the skin depth.

$$\sigma = \frac{2}{\mu_0\omega\delta^2}$$

$$\frac{E_0''}{E_0} = \frac{1 - \frac{c}{\omega\delta} - i\frac{c}{\omega\delta}}{1 + \frac{c}{\omega\delta} + i\frac{c}{\omega\delta}} = \frac{\delta - \frac{c}{\omega} - i\frac{c}{\omega}}{\delta + \frac{c}{\omega} + i\frac{c}{\omega}} \approx -1 + \frac{\omega}{c}\frac{2}{1+i}\delta = -1 + \frac{\omega}{c}(1-i)\delta$$

$$R = \left| \frac{E_0''}{E_0} \right|^2 = (-1 + \delta\omega/c)^2 + \left(\frac{\omega\delta}{c}\right)^2 \approx 1 - 2\delta\omega/c$$