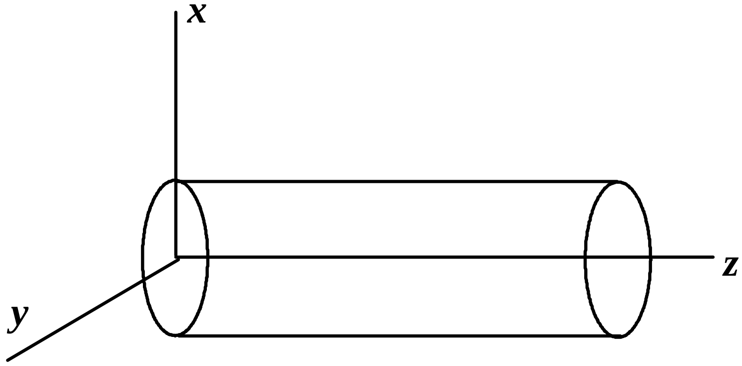
PHY 5347
Homework Set 1 Solutions – Kimel

1. 8.3



a)

$$(\nabla_t^2 + \gamma^2)\psi = 0, \qquad \psi = E_z(TM) \text{ or } \psi = H_z(TE)$$

As in class, we will use cylindrical coordinates, and assume

$$\psi(\rho,\phi) = R(\rho)Q(\phi)$$

We get the two equations

$$\frac{\partial^2}{\partial\phi^2}Q(\phi) = -m^2 Q(\phi) \text{ with solns } Q(\phi) = e^{\pm im\phi},\ m = 0,1,2,\dots$$

$$\frac{d^2}{dx^2}R(x) + \frac{1}{x}\frac{dR(x)}{dx} + \left(1 - \frac{m^2}{x^2}\right)R(x) \quad \left(\text{Bessel eqn.}\right)$$

with regular solutions $J_m(x)$, and singular solution (which we reject as nonphysical) $N_m(x)$. Here $x = \gamma\rho$.

Solutions:

TM: BC: $J_m(x_{mn}) = 0$, and

$$E_z(\rho,\phi) = E_0 J_m(\gamma_{mn}\rho)e^{\pm im\phi}, \quad m = 0,1,2,\dots.;\ n = 1,2,3,\dots;\ \gamma_{mn} = x_{mn}/R$$

Lowest cutoff frequencies:

$$\omega_{mn} = \frac{\gamma_{mn}}{\sqrt{\varepsilon\mu}} = \frac{x_{mn}}{R\sqrt{\varepsilon\mu}}$$

Using the results of Jackson, p. 114,

$$x_{0n} = 2.405, 5.52, 8.654, \ldots$$
$$x_{1n} = 3.832, 7.016, 10.173, \ldots.$$
$$x_{2n} = 5.136, 8.417, 11.620, \ldots.$$

TE: BC: $J'_m(x'_{mn}) = 0,$ and

$$E_z(\rho,\phi) = E_0 J_m(\gamma'_{mn}\rho)e^{\pm im\phi}, \quad m = 0,1,2,\ldots.; \; n = 1,2,3,\ldots; \; \gamma'_{mn} = x'_{mn}/R$$

Lowest cutoff frequencies:

$$\omega_{mn} = \frac{\gamma'_{mn}}{\sqrt{\varepsilon\mu}} = \frac{x'_{mn}}{R\sqrt{\varepsilon\mu}}$$

Using the results of Jackson, p. 370,

$$x'_{0n} = 3.832, 7.016, 10.173, \ldots$$
$$x'_{1n} = 1.841, 5.331, 8.536, \ldots.$$
$$x'_{2n} = 3.054, 6.706, 9.970, \ldots.$$

From the above we see the lowest cutoff frequency is the TE mode

$$\omega'_{11} = 1.841K, \text{ with } K = 1/\left(R\sqrt{\varepsilon\mu}\right)$$

The next four lowest cutoff frequencies are:

$$\omega_{01} = 2.405K = 1.31\omega'_{11}$$
$$\omega'_{21} = 3.054K = 1.66\omega'_{11}$$
$$\omega'_{01} = 3.832K = 2.08\omega'_{11}$$
$$\omega_{11} = 3.832K = 2.08\omega'_{11}$$

b) From Eq. (8.63) in the text

$$\beta_\lambda \propto \left(\frac{\omega}{1-\frac{\omega_\lambda^2}{\omega^2}}\right)^{1/2}\left[\xi_\lambda + \eta_\lambda\left(\frac{\omega_\lambda}{\omega}\right)^2\right]$$

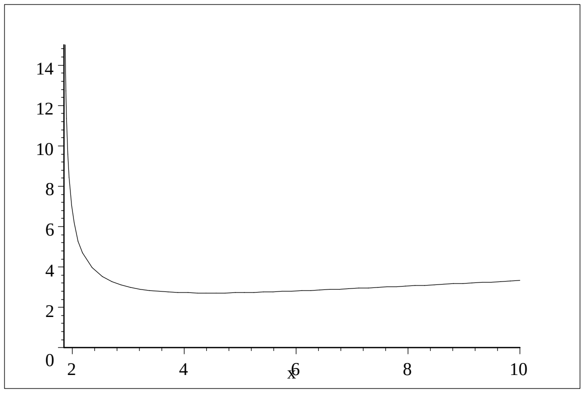
For TM modes, $\eta_\lambda = 0,$ and for TE mode, $\xi_\lambda + \eta_\lambda$ is of order unity. So for comparison purposes, I'll take

$$\beta_{11}(TE) = f_1(x) = \left(\frac{x}{1-\frac{1.841^2}{x^2}}\right)^{1/2}\left(1+\frac{1.841^2}{x^2}\right)$$

$$\beta_{01}(TM) = f_2(x) = \left(\frac{x}{1-\frac{2.405^2}{x^2}}\right)^{1/2}$$

where I've expressed the functions in terms of $x = \omega/K.$

$$\left(1+\frac{1.841^2}{x^2}\right)\left(\frac{x}{1-\frac{1.841^2}{x^2}}\right)^{1/2}$$



$$\left(\frac{x}{1-\frac{2.405^2}{x^2}}\right)^{1/2}$$