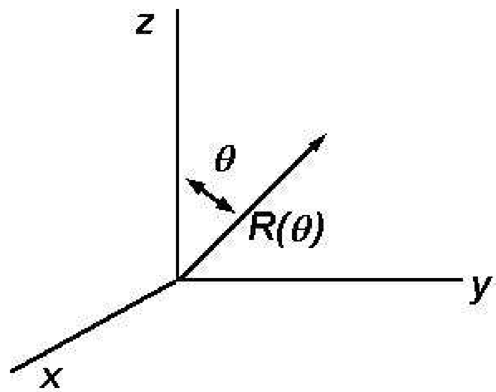
PHY 5347
Homework Set 5 Solutions – Kimel

1. The system is described by



and is azimuthally symmetric

$$R(\theta) = R_0[1+\beta(t)P_2(\cos\theta)]; \quad \beta(t) = \beta_0\cos\omega t; \quad kR << 1$$

$$Q = \int \rho r^2 dr d\phi d\cos\theta = 2\pi\int_{-1}^{1} d\cos\theta\rho\int_0^{R(\theta)} r^2 dr$$

$$= \frac{2\pi}{3}\rho\int_{-1}^{1} R_0^3(1+3\beta P_1 P_2)d\cos\theta + O(\beta^2) = \frac{4\pi}{3}\rho R_0^3 \to \rho = \frac{3}{4\pi R_0^3}Q$$

where I've used the fact that $1 = P_0$. Since the system is azimuthally symmetric, $Q_{lm} = \delta_{m0}Q_{l0}$.

$$Q_{lm} = 2\pi\rho\delta_{m0}\int_{-1}^{1} dxY_l^0\int_0^{R(\theta)} r^{l+2}dr = \frac{2\pi\rho\delta_{m0}}{l+3}\int_{-1}^{1} dxR_0^{l+3}[1+(l+3)\beta P_2]Y_l^0$$

Using $Y_l^0 = \sqrt{\frac{2l+1}{4\pi}}P_l$ and $1 = P_0$,

$$Q_{lm} = \frac{2\pi\rho\delta_{m0}}{l+3}\sqrt{\frac{2l+1}{4\pi}}R_0^{l+3}\left[2\delta_{l0} + (l+3)\beta\frac{2}{2l+1}\delta_{l2}\right]$$

Notice that the $l = 0$ term is time independent and thus does not contribute to the radiation. Next consider the $l = 2$ term.

$$Q_{20}(t) = \frac{2}{5}\sqrt{\pi}\,\rho\sqrt{5}\,R_0^5\beta = \rho = \frac{3}{4\pi R_0^3}Q\frac{2}{5}\sqrt{\pi}\,\rho\sqrt{5}\,R_0^5\beta = \frac{3}{\sqrt{20\pi}}R_0^2 Q\beta(t)$$

$$Q_{20}(t) = \mathrm{Re}\left[\frac{3}{\sqrt{20\pi}}R_0^2 Q\beta_0 e^{-i\omega t}\right]$$

$$Q_{20} = \frac{3}{\sqrt{20\pi}} R_0^2 Q \beta_0$$

$$\frac{dP(2,0)}{d\Omega} = \frac{Z_0}{2k^2} |a(2,0)|^2 \left| \vec{X}_{20} \right|^2$$

$$a_E(2,0) = \frac{ck^4}{i(5 \times 3)} \sqrt{\frac{3}{2}}\, Q_{20}$$

$$\left| \vec{X}_{20} \right|^2 = \frac{15}{8\pi} \sin^2\theta \cos^2\theta$$

$$\frac{dP(2,0)}{d\Omega} = \frac{Z_0}{2k^2} \left| \frac{ck^4}{(5 \times 3)} \sqrt{\frac{3}{2}}\, Q_{20} \right|^2 \times \frac{15}{8\pi} \sin^2\theta \cos^2\theta$$

$$= \frac{1}{160} \frac{Z_0}{k^2} \frac{|ck^4 Q_{20}|^2}{\pi} \sin^2\theta \cos^2\theta = \frac{1}{160} \frac{Z_0}{k^2} \frac{(ck^4)^2 \left(\frac{3}{\sqrt{20\pi}} R_0^2 Q \beta_0 \right)^2}{\pi} \sin^2\theta \cos^2\theta$$

$$= \frac{9}{3200\pi^2} Z_0 k^6 c^2 R_0^4 Q^2 \beta_0^2 \sin^2\theta \cos^2\theta$$

$$P = \frac{9}{3200\pi^2} Z_0 k^6 c^2 R_0^4 Q^2 \beta_0^2 \times 2\pi \int_{-1}^{1} (1 - x^2) x^2 dx = \frac{9}{3200\pi^2} Z_0 k^6 c^2 R_0^4 Q^2 \beta_0^2 \times 2\pi \times \frac{4}{15}$$

$$P = \frac{3}{2000\pi} Z_0 k^6 c^2 R_0^4 Q^2 \beta_0^2$$