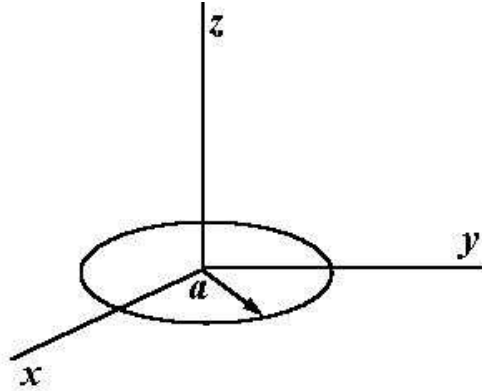


2. The system is described by



$$I(t) = I_0 \cos \omega t = \text{Re}[I_0 e^{-i\omega t}]$$

$$\vec{J}(t) = \frac{1}{a} I(t) \delta(r - a) \delta(\cos \theta) \hat{\phi}$$

where I determined the normalization constant  $\frac{1}{a}$  by the condition  $\int \vec{J} \cdot d\vec{a} = I$

$$\vec{J}(t) = \text{Re} \left[ \frac{I_0}{a} \delta(r - a) \delta(\cos \theta) \hat{\phi} e^{-i\omega t} \right] \rightarrow \vec{J} = \frac{I_0}{a} \delta(r - a) \delta(\cos \theta) \hat{\phi}$$

We use the general expression for  $\vec{H}$  and  $\vec{E}$  in the radiation zone given by Eq.(9.149). Since this system has no net charge density and there is no intrinsic magnetization, the expansion coefficients in these equations are given by

$$a_E(l, m) = \frac{k^2}{i\sqrt{l(l+1)}} \int Y_l^{m*} ik (\vec{r} \cdot \vec{J}) j_l(kr) d^3x$$

$$a_M(l, m) = \frac{k^2}{i\sqrt{l(l+1)}} \int Y_l^{m*} \vec{\nabla} \cdot (\vec{r} \times \vec{J}) j_l(kr) d^3x$$

a)  $\vec{r} \cdot \vec{J} = 0$  in the first equation, so there is no electric multipole radiation. In spherical coordinates

$$\vec{r} \times \vec{J} = -aJ\hat{\theta}$$

Using the formulas for  $\vec{\nabla} \cdot \vec{A}$  in spherical coordinates given in the back of the book,

$$\vec{\nabla} \cdot (\vec{r} \times \vec{J}) = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (J \sin \theta) = -\frac{\cos \theta}{\sin \theta} J - \frac{\partial}{\partial \theta} J$$

The first term does not contribute, because  $\cos \theta = 0$ , while the second term can be written, using the chain rule,

$$\vec{\nabla} \cdot (\vec{r} \times \vec{J}) = \sin\theta \frac{\partial}{\partial \cos\theta} J$$

The problem has azimuthal symmetry, so  $m = 0$ . Realizing derivatives of  $\delta$ -functions are defined by integration by parts,

$$a_M(l, m) = \frac{\delta_{m0} k^2}{i\sqrt{l(l+1)}} \int Y_l^{0*} \left( \sin\theta \frac{\partial}{\partial \cos\theta} J \right) j_l(kr) d^3x = \frac{ik^2}{\sqrt{l(l+1)}} \int \left[ \frac{\partial}{\partial \cos\theta} (\sin\theta Y_l^{0*}) \right] J j_l(kr) d^3x$$

$$a_M(l, 0) = \frac{i2\pi k^2}{\sqrt{l(l+1)}} \frac{I_0}{a} a^2 j_l(ka) \frac{\partial}{\partial \cos\theta} (\sin\theta Y_l^0)|_{\cos\theta=0}$$

$$a_M(l, 0) = \frac{i2\pi k^2}{\sqrt{l(l+1)}} \frac{I_0}{a} a^2 j_l(ka) (1-x^2)^{1/2} \frac{d}{dx} Y_l^0(x)|_{x=0}$$

Since  $Y_l^0(x)$  is either an even or odd polynomial in  $x$ , then only odd  $l$  contribute to  $a_M(l, 0)$ . This determines the expansion coefficients, and thus  $\vec{H}$  and  $\vec{E}$  in the radiation zone are known through Eq.(9.149). The power distribution is given by Eq. (9.151)

b) From our previous answers, we see  $a_E(l, m) = 0$ , and that the lowest magnetic multipole contribution is  $a_M(1, 0)$ .

$$a_M(1, 0) = \frac{i2\pi k^2}{\sqrt{2}} I_0 a j_1(ka) (1-x^2)^{1/2} \frac{d}{dx} Y_1^0(x)|_{x=0}$$

Using

$$j_1(ka) \rightarrow \frac{ka}{3}; \quad \frac{d}{dx} Y_1^0(x)|_{x=0} = \sqrt{\frac{3}{4\pi}}$$

$$a_M(1, 0) = i2\pi k^3 I_0 a^2 \sqrt{\frac{1}{24\pi}} = \frac{ik^3}{3} \sqrt{2} M_{10}$$

$$M_{10} = \frac{i2\pi k^3 I_0 a^2 \sqrt{\frac{1}{24\pi}}}{\frac{ik^3}{3} \sqrt{2}} = \sqrt{\frac{3}{4\pi}} I_0 \pi a^2$$

Note that you would get the same answer, if you used Eq. (9.172) directly.  
From Eq. (9.151)

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2} \left( 2\pi k^3 F a^2 \sqrt{\frac{1}{24\pi}} \right)^2 \frac{3}{8\pi} \sin^2\theta = \frac{1}{32\pi^2} Z_0 k^4 (I_0 \pi a^2)^2 \sin^2\theta$$

If we compare this result with the one that we get for an elementary magnetic dipole, which is given by Eq. (9.23)

with the substitution  $\vec{p} \rightarrow \vec{m}/c$ ,

$$\frac{dP}{d\Omega} = \frac{1}{32\pi^2} Z_0 k^4 |\vec{m}|^2 \sin^2\theta$$

Thus we may identify

$$|\vec{m}| = I_0 \pi a^2$$

as would be expected.