Jackson Problem 1.1
B. J. Mattson

1 Part a

I'm assuming that we are dealing with a conductor at equilibrium. Then when we place excess charge on it. Consider a Gaussian surface with an enclosed charge of q'. In this case:

$$\int \vec{E} \cdot \hat{n}\, da = \frac{1}{\epsilon_0} Q_{enclosed} \tag{1}$$

or $\vec{e} \neq 0$. Since charges are free to move inside a conductor, the excess charge will move to follow the field. But then we are not talking about a conductor in equilibrium. $\Rightarrow\Leftarrow$

So, the charge enclosed in the Gaussian volume must be zero. But the volume chosen was arbitrary, so stretch it just to the edge of the conductor and the enclosed charge must still be zero. the only place left for the excess charge is the surface.

2 Part b

First consider such a conductor with interior charges. Total charge on the original conductor is Q; charge placed in the interior is q.

Take a Guassian surface just outside the surface. Then,

$$\oint \vec{E} \cdot \hat{n} = \frac{1}{\epsilon_0} \int_V \rho(\vec{x}\, dV \tag{2}$$

So,

$$E_\perp A = \frac{Q + q}{\epsilon_0} \tag{3}$$

$$E_\perp = \frac{Q + q}{A\, \epsilon_0} \tag{4}$$

And $E_\perp \neq 0$, so the outside is not protected from those interior charges.

Now consider a conductor with external charges. Take a path from point a on the surface to point b on the surface straight through the conductor. Then use:

$$\oint \vec{E} \cdot \vec{dl} = 0 \tag{5}$$

So,

$$\int_{surfacepath} \vec{E} \cdot \vec{dl} \int_{interiorpath} \vec{E} \cdot \vec{dl} = 0 \tag{6}$$

But the integral over the surface part of the path is 0, since there is no field arbitrarily close to the surface. Then

$$\int_{interiorpath} \vec{E} \cdot \vec{dl} = 0 \tag{7}$$

But this means that, if $E_{interior} \neq 0$,

$$V_a \neq V_b \tag{8}$$

However, a conductor is an equipotential. Therefore, the elecric field on the interior of the sphere must be zero.

3 Part c

Consider a conductor, and a path enclosing a bit of the surface. Now, we know that:

$$\oint \vec{E} \cdot \vec{dl} = 0 \tag{9}$$

Now, squish the side of the path which are perpendicular to the surface of the conductor; then their contribution to the above integral is zero. So,

$$\oint \vec{E} \cdot \vec{dl} = \int_{inside} \vec{E} \cdot \vec{dl} - \int_{outside} \vec{E} \cdot \vec{dl} \tag{10}$$

But $\vec{E} = 0$ inside the conductor, so

$$\oint \vec{E} \cdot \vec{dl} = \int_{outside} \vec{E} \cdot \vec{dl} = 0 \tag{11}$$

Or,

$$E_{\perp outside} l = 0 \tag{12}$$

Since l is arbitrary, $E_{\perp outside} = 0$. Therefore, the electric field just outside of a conductor is perpendicular to the surface of the conducotr.

Now take a Gaussiand pill box enclosing the surface. Squish the side walls, so that they are infimitesimal. Then,

$$\oint_S \vec{E} \cdot \hat{n} = \frac{1}{\epsilon_0} \int_V \rho(\vec{x}) \, d^3x \tag{13}$$

can be written:

$$\oint_{inside} \vec{E} \cdot \hat{n} + \oint_{outside} \vec{E} \cdot \hat{n} = \frac{1}{\epsilon_0} \int_V \rho(\vec{x}) \, d^3x \tag{14}$$

But $\vec{E}_{inside} = 0$. Now squish the box so that $E \sim constant$ over A. Then,

$$\oint_{outside} \vec{E} \cdot \hat{n} = \frac{A \, \sigma}{\epsilon_0} \tag{15}$$

Or,

$$E_{outside} \, A = \frac{A \, \sigma}{\epsilon_0} \tag{16}$$

And $E_{outside} = \frac{\sigma}{\epsilon_0}$