Jackson Problem 6.1
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1 Case 1 - Charge on Surface

So, we have a conductor with charge on the surface.

- $r_1 < a$

$$\oint \vec{E} \cdot \hat{n} = \frac{1}{\epsilon_0} \int \rho(\vec{x}) d^3x \tag{1}$$

Inside the sphere, $\rho = 0$, and $\vec{E}$ is in the $\hat{n}$ direction. So,

$$E \cdot A = 0 \tag{2}$$

$\Rightarrow E = 0$, $r < a$

- $r_2 < a$

$$\oint \vec{E} \cdot \hat{n} = \frac{1}{\epsilon_0} \int \rho(\vec{x}) d^3x \tag{3}$$

So,

$$E \cdot A = \frac{Q}{\epsilon_0} \tag{4}$$

$\Rightarrow E = \frac{Q}{4\pi\epsilon_0 r^2}$, $r > a$

2 Case 2 - Uniform Charge Density

For both inside and out, $\hat{n} = \hat{r}$ and $\vec{E}$.

- $r - 1 < a$

$$\oint \vec{E} \cdot \hat{n} = \frac{1}{\epsilon_0} \int \rho(\vec{x}) d^3x \tag{5}$$

$$E \cdot A = \frac{1}{\epsilon_0} \rho V \tag{6}$$

$$E(4\pi r^2) = frac1\epsilon_0 \rho(\frac{4}{3}\pi r^3) \tag{7}$$

But $\rho = \frac{Q}{\frac{4}{3}\pi a^2}$. So,

$\Rightarrow E = \frac{Q}{4\pi\epsilon_0 a^2} r$, $r < a$

- $r_2 > r$

$$\oint \vec{E} \cdot \hat{n} = \frac{1}{\epsilon_0} \int \rho(\vec{x}) d^3x \tag{8}$$

$$E(4\pi r^2) = \frac{1}{\epsilon_0} Q \tag{9}$$

$\Rightarrow E = \frac{Q}{4\pi\epsilon_0 r^2} r$, $r > a$

3 Case 3 - Non-Uniform Charge Density

Now we have a sphere with charge density, $\rho(r) = \gamma r^n$.

- $r_1 < a$

  Start with the charge density.

$$\begin{aligned}
\int_V \rho(r)\, dV &= \int_0^r \gamma r^n 4\pi r^2 dr && (10)\\
&= 4\pi\gamma \int_0^r r^{n+2} dr && (11)\\
&= 4\pi\gamma \frac{r^n + 3}{n+3} && (12)\\
& \quad n > -3 && (13)\\
& && (14)
\end{aligned}$$

Now to find γ. Set the above integral equal to Q for $r = a$.

$$Q = 4\pi\gamma\frac{a^{n+3}}{n+3} \tag{15}$$

$$\gamma = \frac{Q}{4\pi}\left(\frac{n+3}{a^{n+3}}\right) \tag{16}$$

So,

$$\int_V \rho(r)\, dV = Q\frac{r^{n+3}}{a^{n+3}} \tag{17}$$

And then,

$$\oint \vec{E}\cdot\hat{n} = \frac{1}{\epsilon_0}\int \rho(\vec{x})d^3x \tag{18}$$

$$E(4\pi r^2) = \frac{1}{\epsilon_0}Q\frac{r^{n+3}}{a^{n+3}} \tag{19}$$

$$\tag{20}$$

$\Rightarrow E = \frac{Q}{4\pi\epsilon_0 a^{n+3}}r^{n+1}$, for $r < a$, $n > -3$

- $r > a$

  Outside, the field is the same as the outside field of the previous two cases.

  $\Rightarrow E = \frac{Q}{4\pi\epsilon_0 r^2}r$, $r > a$