Jackson Problem 1.6
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1 Part a

Outside both of the plates, the total $\vec{E}$ field is zero (a Gaussian pillbox including both plates conatins no net charge). Between the plates, the $\vec{E}$ filds from each plate add. So, for the magnitude of the field between the plates, I just need to double the contribution for one plate.

Use a pillbox on the upper plate with area, a, and squish the box so that the contributions from the sides are zero.

$$\oint \vec{E} \cdot \hat{n}\, da = \frac{1}{\epsilon_0} \int \rho(\vec{x})\, d^3x \tag{1}$$

$$E_{outside} a + E_{inside} a = \frac{\sigma a}{\epsilon_0} \tag{2}$$

$$2E = \frac{\sigma}{\epsilon_0} \tag{3}$$

$$\tag{4}$$

So, the field between the plates is $\vec{E} = -\frac{\sigma}{\epsilon_0}\hat{z}$.
Choosing a path in the $-\hat{z}$ direction, the potential difference between the plates is:

$$V = -\int_d^0 \vec{E} \cdot d\vec{l} \tag{5}$$

$$= \int_0^d \frac{\sigma}{\epsilon_0} dl \tag{6}$$

$$= \frac{\sigma}{\epsilon_0} d \tag{7}$$

$$\tag{8}$$

But, $\sigma = \frac{Q}{A}$, so $V = \frac{Qd}{\epsilon_0 A}$.
And, the capcitance is:
$\Rightarrow C = \frac{Q}{V} = \frac{\epsilon_0 A}{d}$

2 Part b

The field between a and b is:

$$\oint \vec{E} \cdot \hat{n}\, da = \frac{1}{\epsilon_0} \int \rho(\vec{x})\, d^3x \tag{9}$$

$$4\pi r^2 E = \frac{Q}{\epsilon_0} \tag{10}$$

$$E = \frac{Q}{4\pi\epsilon_0 r^2} \tag{11}$$

$$\tag{12}$$

Then,

$$V = -\int_b^a \vec{E} \cdot \vec{dl} \tag{13}$$

$$= -\frac{Q}{4\pi\epsilon_0} \int_b^a r^{-2} dr \tag{14}$$

$$= \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{a} - \frac{1}{b}\right] \tag{15}$$

$$\tag{16}$$

And the capcitance is:
$\Rightarrow C = \frac{Q}{V} = 4\pi\epsilon_0 \frac{ab}{b-a}$

3 Part c

Using a cylinder between the conducting cylinders as the Gaussian surface,

$$\oint \vec{E} \cdot \hat{n}\, da = \frac{1}{\epsilon_0} \int \rho(\vec{x})\, d^3x \tag{17}$$

$$2\pi r L E = \frac{Q}{\epsilon_0} \tag{18}$$

$$E = \frac{Q}{2\pi L \epsilon_0 r} \tag{19}$$

$$\tag{20}$$

So the potential is:

$$
\begin{aligned}
V &= -\int_b^a \vec{E} \cdot \vec{dl} && (21)\\
&= -\frac{Q}{2\pi L\epsilon_0}\int_b^a r^{-1} dr && (22)\\
&= \frac{Q}{2\pi L\epsilon_0}(ln(b) - ln(a)) && (23)\\
&= \frac{Q}{2\pi L\epsilon_0} ln\left(\frac{b}{a}\right) && (24)\\
& && (25)
\end{aligned}
$$

And the capacitance:
$\Rightarrow C = \frac{Q}{V} = 2\pi L\epsilon_0 \left[ln\left(\frac{b}{a}\right)\right]^{-1}$