Jackson Problem 1.12
B. J. Mattson

Start with Green's theorem:

$$\int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\hat{n}\,da = \int_V dV[\phi\nabla^2\psi - \psi\nabla^2\phi] \tag{1}$$

Now, let $\phi = \Phi$ and $\psi = \Phi'$. Then,

$$\int_S (\Phi\nabla\Phi' - \Phi'\nabla\Phi)\cdot\hat{n}\,da = \int_V dV[\Phi\nabla^2\Phi' - \Phi'\nabla^2\Phi] \tag{2}$$

Or,

$$\int_V dV\,\Phi'\nabla^2\Phi - \int_S (\Phi'\nabla\Phi)\cdot\hat{n}\,da = \int_V dV\,\Phi\nabla^2\Phi' - \int_S (\Phi\nabla\Phi')\cdot\hat{n}\,da \tag{3}$$

Now, since Φ and Φ' are potentials, they satisfy $\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$ and $\nabla^2\Phi' = -\frac{\rho'}{\epsilon_0}$.

As was shown in problem 1, part c, the $\vec{E}$ field is perpendicular to a conducting surface. Also, $\vec{E}$ and $\hat{n}$ are anti-parallel, so

$$\begin{aligned} \nabla\Phi\cdot\hat{n} &= -\vec{E}\cdot\hat{n} && (4)\\ &= \frac{\sigma}{\epsilon_0} && (5)\\ \nabla\Phi'\cdot\hat{n} &= -\vec{E}\cdot\hat{n} && (6)\\ &= \frac{\sigma'}{\epsilon_0} && (7)\\ & && (8) \end{aligned}$$

Using these relations in 3,

$$-\int_V dV\,\Phi'\frac{\rho}{\epsilon_0} - \int_S \Phi'\frac{\sigma}{\epsilon_0}da = -\int_V dV\,\Phi\frac{\rho}{\epsilon_0} - \int_S \Phi\frac{\sigma}{\epsilon_0}da \tag{9}$$

So,

$$\int_V \Phi'\rho\,dV + \int_S \Phi'\sigma da = \int_V \Phi\rho\,dV + \int_S \Phi\sigma' da \tag{10}$$