Jackson Problem 2.15
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1 Part a

In general, the Green function must satisfy $\nabla'^2 G_D = -4\pi\delta(\vec{x}-\vec{x'})$. For Dirichlet boundary conditions, the Green function has boundary conditions: $G(boundaries) = 0$. So I need to solve:

$$\nabla'^2 G_D = -4\pi\delta(x-x')\delta(y-y') \tag{1}$$

with $G_D(x'=0) = G_D(x'=1) = G_D(y'=0) = G_D(y'=1) = 0$.

Assume $G_D = g_x(x,x')g_y(y,y')$. Then the equation can be written:

$$g_y\frac{\partial^2 g_x}{\partial x'^2} + g_x\frac{\partial^2 g_y}{\partial y'^2} = -4\pi\delta(x-x')\delta(y-y') \tag{2}$$

Look at $x \neq x'$ and $y \neq y'$. Then,

$$\frac{1}{g_x}\frac{\partial^2 g_x}{\partial x'^2} + \frac{1}{g_y}\frac{\partial^2 g_y}{\partial y'^2} = 0 \tag{3}$$

And, since there is no x' or y' depencence on the RHS,

$$\begin{aligned}\frac{\partial^2 g_x}{\partial x'^2} &= -\pi^2 n^2 g_x &(4)\\ \frac{\partial^2 g_y}{\partial y'^2} &= \pi^2 n^2 g_y &(5)\\ & &(6)\end{aligned}$$

Solving for g_x,

$$g_x(x,x') = \sin(n\pi x)\sin(n\pi x') \tag{7}$$

Now let (leaving the $= 4\pi$ with the $\delta(y-y')$),

$$\delta(x-x') = \sum_{n=1}^{\infty} c_n \sin(n\pi x)\sin(n\pi x') \tag{8}$$

And c_n can be found using orthoganality considerations,

$$\begin{aligned}\int_0^1 \delta(x-x')\sin(n\pi x')dx' &= c_n sin(n\pi x)\cdot\frac{1}{2} &(9)\\ \sin(n\pi x) &= c_n sin(n\pi x)\cdot\frac{1}{2} &(10)\\ & &(11)\end{aligned}$$

So $c_n = 2$.
And

$$g_x = 2\sum_{n=1}^{\infty} \sin(n\pi x)\sin(n\pi x') \tag{12}$$

So,

$$G(x,x',y,y') = 2\sum_{n=1}^{\infty} g_n(y,y')\sin(n\pi x)\sin(n\pi x') \tag{13}$$

Where g_n satisfies

$$\frac{\partial^2 g_n}{\partial y'^2} - \pi^2 n^2 g_n = -4\pi\delta(y-y') \tag{14}$$

2 Part b

To satisfy the boundary conditions ($g_n(y,0) = g_n(y,1) = 0$), let

$$g_n = A\left\{ \begin{array}{ll} \sinh(\pi n y')\sinh[\pi n(1-y)] & y' < y \\ \sinh[\pi n(1-y')]\sinh(\pi n y) & y < y' \end{array} \right.$$

This satisfies both the boundary conditions and one of the jump conditions. However, I still need A, which I can find by using the discontinuity in slope.

$$\left.\frac{\partial g_n}{\partial y'}\right|_{y-}^{y+} = -4\pi \tag{15}$$

Or,

$$A[\pi n\cosh(\pi n y)\sinh[\pi n(1-y)] - \pi n\sinh(\pi n y)\cosh[\pi n(1-y)] = 4\pi \tag{16}$$

After a bit of algebra,

$$\begin{aligned} A &= \frac{4}{n}\left[\cosh(\pi n y)\left[\sinh(\pi n)\cosh(-\pi n y) - \cosh(\pi n)\sinh(-\pi n y)\right] - \right. && (17) \\ & \left.\sinh(\pi n y)\left[\cosh(\pi n)\cosh(-\pi n y) - \sinh(\pi n)\sinh(-\pi n y)\right]\right]^{-1} && (18) \end{aligned}$$

And a bit more magic (hyperbolic identities, etc.) gives:

$$A = \frac{4}{n\sinh(\pi n)} \tag{19}$$

So,

$$g_n = \frac{4}{n \sinh(\pi n)} \sinh(\pi n y_<) \sinh[\pi n(1 - y_>)] \tag{20}$$

Finally, plugging this into the earlier expression for $G(x, y, x', y')$,

$$G(x, y, x', y') = 8 \sum_{n=1}^{\infty} \frac{1}{n \sinh(\pi n)} \sin(\pi n x) \sin(\pi n x') \sinh(\pi n y_<) \sinh[\pi n(1 - y_>)] \tag{21}$$