Jackson Problem 3.6
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1 Part a

Let $\vec{x_1} = (a, 0, 0)$ and $\vec{x_2} = (a, \pi, 0)$ (in cylindrical coordinates). Then the potential at $\vec{x}$ is just the superposition of the potentials of the two charges.

$$\Phi(\vec{x}) = \frac{\frac{q}{4\pi\epsilon_0}}{|\vec{x} - \vec{x_1}|} - \frac{\frac{q}{4\pi\epsilon_0}}{|\vec{x} - \vec{x_2}|} \tag{1}$$

Writing $|\vec{x} - \vec{x'}|$ in terms of spherical harmonics:

$$\Phi(\vec{x}) = \frac{q}{4\pi\epsilon_0}\left[4\pi\sum_{l,m}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}\left(Y_{lm}^*(0,\phi_1)Y_{lm}(\theta,\phi) - Y_{lm}^*(\pi,\phi_2)Y_{lm}(\theta,\phi)\right)\right] \tag{2}$$

The problem is azimuthally symmetric, so only the $m = 0$ terms survive the sum. Also,

$$Y_{lm}^*(0,\phi_1) = \sqrt{\tfrac{2l+1}{4\pi}}P_l(\cos 0) = \sqrt{\frac{2l+1}{4\pi}} \tag{3}$$

$$Y_{lm}^*(\pi,\phi_2) = \sqrt{\tfrac{2l+1}{4\pi}}P_l(\cos\pi) = (-1)^l\sqrt{\frac{2l+1}{4\pi}} \tag{4}$$

So the potential can be rewrittenn:

$$\Phi(\vec{x}) = \frac{q}{\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\sqrt{\frac{2l+1}{4\pi}}\frac{r_<^l}{r_>^{l+1}}Y_{l0}(\theta,\phi)[1-(-1)^l] \tag{5}$$

From the last term, only the odd l survive, so using the relationship between Y_{l0} and P_l,

$$\Phi(r < a, \theta, \phi) = \frac{q}{2\pi\epsilon_0}\sum_{odd\, l}\frac{a^l}{r^{l+1}}P_l(\cos\theta) \tag{6}$$

$$\Phi(r < a, \theta, \phi) = \frac{q}{2\pi\epsilon_0}\sum_{odd\, l}\frac{r^l}{a^{l+1}}P_l(\cos\theta) \tag{7}$$

2 Part b

To keep qa constant for $a \to 0$, let $q \to \infty$ and $a = \frac{p}{2q}$. Then

$$\Phi(r > a, \theta, \phi) = \frac{q}{2\pi\epsilon_0} \sum_{l\, odd} \left(\frac{p}{2q}\right)^l \frac{1}{r^{l+1}} P_l(\cos\theta) \tag{8}$$

$$= \frac{1}{2\pi\epsilon_0} \sum_{l\, odd} \frac{p^l}{2^l q^{l-1}} \frac{1}{r^{l+1}} P_l(\cos\theta) \tag{9}$$

So, for $q \to \infty$, $\frac{1}{q^{l-1}}$ will die for all $l > 1$. Then rewritting with just the $l = 1$ term,

$$\Phi(a \to 0, qa = \frac{p}{2}) = \frac{p}{4\pi\epsilon_0 r^2} \cos\theta \tag{10}$$

3 Part c

The charge distribution is

$$\rho(\vec{x}) = q\delta(r - a)[\delta(\cos\theta - 1) - \delta(\cos\theta + 1)] \tag{11}$$

Then,

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3\vec{x'} \rho(\vec{x}) G_D(\vec{x}, \vec{x'}) \tag{12}$$

$$= \frac{q}{\epsilon_0} \int d(\cos\theta') \int d\phi \int r'^2 dr' \delta(r' - a)[\delta(\cos\theta' - 1) - \delta(\cos\theta' + 1)] \tag{13}$$

$$\sum_{l,m} \frac{1}{2l+1} \left[\frac{r_<^l}{r_>^{l+1}} - \frac{(r'r)^l}{b^{2l+1}}\right] Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$$

Using azimuthal symmetry and performing the ϕ' integration:

$$\Phi(\vec{x}) = \frac{q}{2\epsilon_0} \int d(\cos\theta') \int r'^2 dr' \delta(r' - a)[\delta(\cos\theta' - 1) - \delta(\cos\theta' + 1)] \tag{14}$$

$$\sum_{l} \left[\frac{r_<^l}{r_>^{l+1}} - \frac{(r'r)^l}{b^{2l+1}}\right] P_l(\cos\theta') P_l(\cos\theta)$$

Using the delta functions to perform the remaining integrals:

$$\Phi(\vec{x}) = \frac{q}{2\epsilon_0} a^2 \sum_{l=0}^{\infty} \left[\frac{r_<^l}{r_>^l} - \frac{(ra)^l}{b^{2l+1}}\right] P_l(\cos\theta) \left[P_l(1) - P_l(-1)\right] \tag{15}$$

Only the odd l survive (as in Part a). So

$$\Phi(r, \theta, \phi) = \frac{qa^2}{\epsilon_0} \sum_{l\ odd}^{\infty} \left[\frac{r_<^l}{r_>^l} - \frac{(ra)^l}{b^{2l+1}} \right] P_l(\cos\theta) \tag{16}$$