Jackson Problem 4.2
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Let's just try stuffing the given ρ into the Φ and ω calculations and see what pops out. Note: From Arfken section 1.15

$$\int f(x') \cdot \nabla' \delta(x' - x_0)\, dx = -\nabla f(x_0) \tag{1}$$

Potential calculation:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x'})}{|\vec{x} - \vec{x'}|} d^3x' \tag{2}$$

$$= -\frac{1}{4\pi\epsilon_0} \int \frac{\vec{p} \cdot \nabla' \delta(\vec{x'} - \vec{x_0})}{|\vec{x} - \vec{x'}|} d^3x' \tag{3}$$

Now, $\vec{p}$ is not a function of $\vec{x'}$, so I can pull this out of the integral and do the dot product later.

$$\Phi(\vec{x}) = -\frac{1}{4\pi\epsilon_0} \vec{p} \cdot \int \frac{\nabla' \delta(\vec{x'} - \vec{x_0})}{|\vec{x} - \vec{x'}|} d^3x' \tag{4}$$

$$= \frac{1}{4\pi\epsilon_0} \vec{p} \cdot \left(\nabla' \frac{1}{|\vec{x} - \vec{x'}|} |_{\vec{x'} = x_0} \right) \tag{5}$$

$$= \frac{1}{4\pi\epsilon_0} \vec{p} \cdot \left(\frac{\vec{x} - \vec{x_0}}{|\vec{x} - \vec{x_0}|^3} \right) \tag{6}$$

Which just happens to be the potential of a dipole at $\vec{x_0}$.
Energy calculation:

$$W = \int \rho(\vec{x'}) \Phi(\vec{x'})\, d^3x' \tag{7}$$

$$= -\int \vec{p} \cdot \nabla' \delta(\vec{x'} - \vec{x_0}) \Phi(\vec{x'})\, d^3x' \tag{8}$$

Again, I can pull the $\vec{p}$ out of the integral.

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\begin{aligned}
W &= -\vec{p} \cdot \int \nabla' \delta(\vec{x'} - \vec{x_0}) \Phi(\vec{x'})\, d^3x' & (9) \\
&= \vec{p} \cdot \left(\nabla' \Phi(\vec{x'})|_{\vec{x'}=\vec{x_0}} \right) & (10) \\
&= -\vec{p} \cdot \vec{E}(\vec{x_0}) & (11)
\end{aligned}
$$

Which is the energy of a dipole at $\vec{x_0}$.