Jackson Problem 4.10
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Note that I chose the $z-$axis to be toward the right in the figure given in *Jackson*. Also I defined the following regions: A is the region inside the spheres; B is the region between the spheres filled with the dielectric (ϵ); C is the other region between the spheres (ϵ_0); and D is the region outside the spheres.

1 Parts a and b

I did this in full gory detail. The general expantion of Φ is

$$\Phi(\vec{x}) = \sum_{lm} Y_{lm}(\theta,\phi)\left[A_l r^l + B_l r^{-(l+1)}\right] \tag{1}$$

Or, from azimuthal symmetry,

$$\Phi(\vec{x}) = \sum_{l} P_l(\cos\theta)\left[A_l r^l + B_l r^{-(l+1)}\right] \tag{2}$$

This needs to be true for all $\cos\theta$. In all regions, $\Phi(\vec{x}) \neq \Phi(\cos\theta)$
Now look at each region:

- A: $r < a$

$$\Phi(\vec{x}) = \sum_{l} P_l(\cos\theta)\left[A_l r^l\right] \tag{3}$$

From $\Phi(\vec{x}) \neq \Phi(\cos\theta)$, only the $l = 0$ term survives. So,

$$\Phi(\vec{x_0}) = A \tag{4}$$

- B and C: $a < r < b$

$$\Phi(\vec{x}) = \sum_{l} P_l(\cos\theta)\left[B_l r^l + C_l r^{-(l+1)}\right] \tag{5}$$

Again, only the $l = 0$ term survives.

$$\Phi(\vec{x}) = B + \frac{C}{r} \tag{6}$$

- D: $r > b$

$$\Phi(\vec{x}) = \sum_l P_l(\cos\theta)\left[D_l r^{-(l+1)}\right] \tag{7}$$

Again, only the $l = 0$ term survives.

$$\Phi(\vec{x}) = \frac{D}{r} \tag{8}$$

Then if we use the boundary conditions that Φ is continuous everywhere,

$$A = B + \frac{C}{a} \tag{9}$$

$$B + \frac{C}{b} = \frac{D}{b} \tag{10}$$

Now consider the $\vec{D}$ boundary conditions:

$$\vec{E}_{(B\ or\ C)} = -\nabla\Phi = \frac{C}{r^2}\hat{r} \tag{11}$$

$$\vec{E}_{(D)} = -\nabla\Phi = \frac{C + Bb}{r^2}\hat{r} \tag{12}$$

Then

$$\nabla \cdot \vec{D}_{(BD)} = \sigma_{free} \tag{13}$$

$$\epsilon_0 \frac{C + Bb}{b^2} - \epsilon\frac{C}{b^2} = \sigma_{BD} \tag{14}$$

$$\nabla \cdot \vec{D}_{(CD)} = \sigma_{free} \tag{15}$$

$$\epsilon_0 \frac{C + Bb}{b^2} - \epsilon_0\frac{C}{b^2} = \sigma_{CD} \tag{16}$$

$$= \epsilon\frac{B}{b} \tag{17}$$

$$\tag{18}$$

Or, for the inner edge/boundary,

$$\nabla \cdot \vec{D}_{(AB)} = \sigma_{AB} \tag{19}$$

$$\sigma_{AB} = \epsilon\frac{C}{a^2} - 0 \quad (20)$$
$$\nabla\cdot\vec{D}_{(AC)} = \sigma_{AC} \quad (21)$$
$$\sigma_{AC} = \epsilon_0\frac{C}{a^2} - 0 \quad (22)$$
$$(23)$$

And the total effective charge on the inner sphere is just

$$Q = 2\pi a^2[\sigma_{AC} + \sigma_{AB}] \quad (24)$$
$$= 2\pi a^2[\epsilon\frac{C}{a^2} + \epsilon_0\frac{C}{a^2} \quad (25)$$

So the constant, C is

$$C = \frac{Q}{2\pi(\epsilon+\epsilon_0)} \quad (26)$$

And between the spheres, $\vec{E}$ is

$$\vec{E} = \frac{Q}{2\pi r^2(\epsilon+\epsilon_0)}\hat{r} \quad (27)$$

The surface charge density induced on the inner sphere (part b) is just:

$$\sigma_{AB} = \frac{Q\epsilon}{2\pi a^2(\epsilon+\epsilon_0)} \quad (28)$$
$$\sigma_{AC} = \frac{Q\epsilon_0}{2\pi a^2(\epsilon+\epsilon_0)} \quad (29)$$

2 Part c

Now

$$\sigma_{pol} = -(\vec{P}_2 - \vec{P}_1)\cdot\hat{n_{21}} \quad (30)$$

And $P_i = (\epsilon_i - \epsilon_0)\vec{E}_i$
But $P_1 = P_A = (\epsilon_0 - \epsilon_0)\cdot 0 = 0$
Then,

$$P_0 = (\epsilon - \epsilon_0)\vec{E}_B \quad (31)$$
$$= \frac{Q}{2\pi a^2}\left(\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\right)\hat{r} \quad (32)$$

And, finally, $\hat{n} = -\hat{r}$

$$\sigma_{pol} = \frac{Q}{2\pi a^2} \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \right) \tag{33}$$