Jackson Problem 5.4
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1 Part a

Since I'm considering a current free region,

$$\vec{\nabla} x \vec{B} = \mu_0 J = 0 \tag{1}$$

And I can define a magnetic scalar potential

$$\vec{B} = -\nabla \Phi_{mag} \tag{2}$$

Then I just need to solve Laplace's equation

$$\nabla^2 \Phi_{mag} = 0 \tag{3}$$

In a cylindrical region. But I've done this for electrostatics. The solution is just,

$$\Phi_m = R(\rho) Z(z) Q(\phi) \tag{4}$$

where

$$\begin{aligned} R(\rho) &= A J_\nu(k\rho) + B N_\nu(k\rho) & (5) \\ Z(z) &= C e^{-kz} & (6) \\ Q(\phi) &= D e^{i\nu\phi} & (7) \end{aligned}$$

However, since Φ_m must be well behaved for $\rho \to 0$, $B = 0$. Also, we're given that the magnetic field (and hence Φ_m) is not a function of ϕ, so $\nu = 0$.

Gathering all the constants together,

$$\Phi_{mag} = C J_0(k\rho) e^{-kz} \tag{8}$$

And

$$\begin{aligned} \vec{B} &= -\nabla \Phi_{mag} & (9) \\ &= -\left[\hat{\rho} \frac{\partial \Phi_{mag}}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial \Phi_{mag}}{\partial \phi} + \hat{z} \frac{\partial \Phi_{mag}}{\partial z} \right] & (10) \\ &= -\hat{\rho} C k \frac{\partial J_0(x)}{\partial x} e^{-kz} + \hat{z} C k J_0(k\rho) e^{-kz} & (11) \end{aligned}$$

However, $\frac{\partial J_0(x)}{\partial x} = -J_1(x)$. So,

$$\vec{B} = \hat{\rho} C k J_1(k\rho) e^{-kz} + \hat{z} C k J_0(k\rho) e^{-kz} \tag{12}$$

And we have

$$\begin{aligned} B_\rho(\rho, z) &= CkJ_1(k\rho)e^{-kz} && (13) \\ B_z(\rho, z) &= CkJ_0(k\rho)e^{-kz} && (14) \\ & && (15) \end{aligned}$$

Also, $B_z(0, z) = Cke^{-kz}$, so $\frac{\partial^n B_z(0,z)}{\partial z^n}$.

Now expand J_1 and J_0.

$$\begin{aligned} B_\rho(\rho, z) &\approx Cke^{-kz}\left(\frac{k\rho}{2} - \frac{k^3\rho^3}{16} + \ldots\right) && (16) \\ &\approx -\left(\frac{\rho}{x}\right)\frac{\partial B_z(0,z)}{\partial z} + \left(\frac{\rho^3}{16}\right)\frac{\partial^3 B_z(0,z)}{\partial z^3} + \ldots && (17) \end{aligned}$$

And

$$\begin{aligned} B_\rho(\rho, z) &\approx Cke^{-kz}\left(1 - \frac{k^2\rho^2}{4} + \ldots\right) && (18) \\ &\approx B_z(0, z) - \left(\frac{\rho^2}{4}\right)\frac{\partial^2 B_z(0,z)}{\partial z^2} + \ldots && (19) \end{aligned}$$

2 Part b

In general, the criterion for truncating a sum is that the ratio of what would've been the next term to the last kept term is much less than one.

Take the B_z series and require that

$$\frac{Cke^{-kz}\frac{k^4\rho^4}{2^2 4^2}}{Cke^{-kz}\frac{k^2\rho^2}{2^2}} << 1 \tag{20}$$

Or

$$\rho^2 << \frac{16}{k^2} \tag{21}$$

Now try the B_ρ series:

$$\frac{Cke^{-kz}\frac{k^5\rho^5}{2^2 4^2 6}}{Cke^{-kz}\frac{k^3\rho^3}{2^2 4}} << 1 \tag{22}$$

Or

$$\rho^2 << \frac{24}{k^2} \tag{23}$$

But the first restriction was tighter, so "near" the axis would be defined by

$$\rho^2 << \frac{16}{k^2} \tag{24}$$