Jackson Problem 5.13
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First, cleverly align the spin diameter with the z-axis of our coodinate system. Also let $\vec{r'}$ be a vector from the center of the sphere to a surface charge element. Then,

$$\begin{aligned}\vec{J} &= \sigma\vec{v} \\ &= \sigma\left(\vec{\omega}x\vec{r'}\right) \quad (1)\\ &= \sigma\omega\delta(r'-a)\hat{\phi}' \quad (2)\end{aligned}$$

Now, since the charge distribution is uniform, it shouldn't matter at which ϕ I observe. I'll choose a convenient one...let $\phi = 0$.

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x'})}{|\vec{x}-\vec{x'}|}d^3x' \quad (3)$$

Now, $\vec{J} = \vec{J}_\phi\hat{\phi}$, so $\vec{A}$ only has a ϕ component. Also, $\hat{\phi}' = \sin\theta'\cos\theta'\hat{\jmath}$.

$$\begin{aligned}A_\phi &= \frac{\mu_0}{4\pi}\int [\omega\sigma\delta(r'-a)\sin\theta'\cos\theta']\left[4\pi\sum_{l,m}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,0)\right] \quad (4)\\ &= \omega\sigma\mu_0\int\sum_{l,m}\frac{1}{2l+1}Y_{lm}(\theta,0)\\ &\quad Re\left(\int r'^3\sin^2\theta' e^{i\phi'}\delta(r'-a)\frac{1}{2l+1}Y_{lm}^*(\theta',\phi')dr'\,d\theta'\,d\phi'\right) \quad (5)\end{aligned}$$

The presence of the $e^{i\phi'}$ means that only the $m=1$ term will survive the ϕ' integration. Then, using the definition of Y_{l1}^* (and Y_{l1}) in terms of P_l^1,

$$A_\phi(\vec{x}) = \frac{\omega\sigma\mu_0}{2}\sum_l\frac{1}{l(l+1)}P_l^1(cos\theta)\left[\int r'^3\frac{r_<^l}{r_>^{l+1}}\delta(r'-a)\,dr'\right]\left[\int\sin^2\theta'\,P_l^1(cos\theta')\,d\theta'\right] \quad (6)$$

Looking at the ϕ' integration (I_θ):

$$\begin{aligned}I_\theta &= \int_0^\pi \sin^2\theta'\,P_.^1(\cos\theta')d\theta' \quad (7)\\ &= \int_{-1}^1 (1-x^2)^{\frac{1}{2}}P_l^1(x)\,dx \quad (8)\end{aligned}$$

But,

$$P_l^1 = (-1)^1(1-x)^{\frac{1}{2}}\frac{d}{dx}P_l(x) \quad (9)$$

So,

$$
\begin{aligned}
I_\theta &= -\int_{-1}^{1}(1-x^2)\frac{d}{dx}P_l(x)dx && (10)\\
&= \int_{-1}^{1}\frac{l(l+1)}{2l+1}\left[P_{l+1}(x)-P_{l-1}(x)\right]dx && (11)\\
&= \frac{l(l+1)}{2l+1}\left[\frac{1}{2(l+1)+1}\left(P_{l+2}(x)+P_l(x)\right)-\frac{1}{2(l+1)+1}\left(P_l(x)-P_{l-2}(x)\right)\right]_{-1}^{1} && (12)
\end{aligned}
$$

Where that last step came from recursion relations for P_l. However, from the normalization of P_l, $P_l(\pm1) = P_{l+2}(\pm1)$. So, $I_\theta = 0$ for $l \geq 2$. Then, I just need I_θ for $l = 0, 1$. Working these out from the definition of P_0 and P_1,

$$
\begin{aligned}
I_\theta &= 0 && (13)\\
I_\theta &= -\frac{4}{3} && (14)
\end{aligned}
$$

Using this, the fact that only the $l = 1$ term survives the sum and $P_1^1 = (\cos\theta) = -(1-\cos^2\theta)^{\frac{1}{2}}\frac{d}{d(\cos\theta)}\cos\theta = -\sin\theta$. Then

$$
A_\phi(\vec{x}) = \frac{\omega\sigma\mu_0}{3}\sin\theta\int r'^3\frac{r_<}{r_>^2}\delta(r'-a)dr' \quad (15)
$$

And,

$$
\begin{aligned}
\vec{A}(\vec{x}; r<a) &= \frac{\omega\sigma\mu_0}{3}ar\sin\theta\hat{\phi} && (16)\\
\vec{A}(\vec{x}; r>a) &= \frac{\omega\sigma\mu_0}{3}a^4\frac{\sin\theta}{r^2}\hat{\phi} && (17)
\end{aligned}
$$

Then, to find $\vec{B}$, $\vec{B} = \vec{\nabla}x\vec{A}$,

$$
\begin{aligned}
\vec{B}(\vec{x}, r<a) &= \frac{\omega\sigma\mu_0 a}{3}\left[2\cos\theta\hat{r} - 2\sin\theta\hat{\theta}\right] && (18)\\
\vec{B}(\vec{x}, r>a) &= \frac{\omega\sigma\mu_0 a^4}{3}\left[\frac{2\cos\theta}{r^3}\hat{r} - \frac{2\sin\theta}{r^3}\hat{\theta}\right] && (19)
\end{aligned}
$$