Jackson Problem 5.33
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1 Part a

From Jackson, Eqn (5.10),

$$\vec{F_{12}} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{\vec{dl_1} \cdot \vec{dl_2}}{|\vec{x_{12}}|^3} \vec{x_{12}} \tag{1}$$

From the geometry of the loops,

$$\vec{x_{12}} = \vec{x_1} - \vec{x_2} + \vec{R} \tag{2}$$

So,

$$\vec{F_{12}} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \left(\vec{dl_1} \cdot \vec{dl_2}\right) \frac{\vec{x_1} - \vec{x_2} + \vec{R}}{|\vec{x_1} - \vec{x_2} + \vec{R}|^3} \tag{3}$$

$$= -\frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \left(\vec{dl_1} \cdot \vec{dl_2}\right) \vec{\nabla}_R \frac{-1}{|\vec{x_1} - \vec{x_2} + \vec{R}|} \tag{4}$$

$$= I_1 I_2 \vec{\nabla}_R \left(\frac{\mu_0}{4\pi} \oint \oint \left(\vec{dl_1} \cdot \vec{dl_2}\right) \frac{1}{|\vec{x_1} - \vec{x_2} + \vec{R}|} \right) \tag{5}$$

The $\vec{\nabla}_R$ can be pulled out of the integral, since it is not a function of l_1 or l_2. Now, define

$$M_{12}(\vec{R}) = \frac{\mu_0}{4\pi} \oint \oint \frac{\vec{dl_1} \cdot \vec{dl_2}}{|\vec{x_1} - \vec{x_2} + \vec{R}|} \tag{6}$$

And finally we have,

$$\vec{F_{12}} = I_1 I_2 \vec{\nabla}_R M_{12}(\vec{R}) \tag{7}$$

2 Part b

By our previous definition,

$$\vec{\nabla}_R^{\,2} M_{12}(\vec{R}) = \vec{\nabla}_R^{\,2} \frac{\mu_0}{4\pi} \oint \oint \frac{\vec{dl_1} \cdot \vec{dl_2}}{|\vec{x_1} - \vec{x_2} + \vec{R}|} \tag{8}$$

$$= \frac{\mu_0}{4\pi} \oint \oint (\vec{dl_1} \cdot \vec{dl_2}) \vec{\nabla}_R^{\,2} \frac{1}{|\vec{x_1} - \vec{x_2} + \vec{R}|} \tag{9}$$

Now let $\vec{x} = \vec{x_1} - \vec{x_2}$ and $\vec{x'} = \vec{R}$, then $\vec{\nabla}_R{}^2 \rightarrow \vec{\nabla'}^2$. And

$$\vec{\nabla}_R{}^2 M(\vec{R}) = \frac{\mu_0}{4\pi} \oint \oint (d\vec{l_1} \cdot d\vec{l_2}) \vec{\nabla'}^2 \frac{1}{|\vec{x} - \vec{x'}|} \tag{10}$$

However, we already know that $\frac{1}{|\vec{x}-\vec{x'}|}$ is a solution of Laplace's equation. I.e. $\vec{\nabla'}^2 \frac{1}{|\vec{x}-\vec{x'}|} = 0$ So,

$$\vec{\nabla}_R{}^2 M_{12}(\vec{R}) = 0 \tag{11}$$