Jackson Problem 6.1
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1 Part a

In general, the solution will be,

$$\psi(\vec{x},t) = \int \frac{[f(\vec{x'},t')]_ret}{|\vec{x}-\vec{x'}|} d^3x' \tag{1}$$

So, using the given f,

$$\psi(\vec{x},t) = \int \frac{\delta(x')\delta(y')\delta\left(t - \frac{|\vec{x}-\vec{x'}|}{c}\right)}{|\vec{x}-\vec{x'}|} dx'dy'dz' \tag{2}$$

The x' and y' integrations are easy enough,

$$\psi(\vec{x},t) = \int dz' \frac{\delta\left(t - \frac{\sqrt{x^2+y^2+(z-z')^2}}{c}\right)}{\sqrt{x^2+y^2+(z-z')^2}} \tag{3}$$

Then, using the formula $\int dx \delta\left(f(x)\right) g(x) = \sum_{x_0} \frac{g(x_0)}{|f'(x_0)|}$, where the x_0 are defined by $f(x_0) = 0$,

$$\begin{aligned} \psi(\vec{x},t) &= \sum_{z_0} \left[\frac{1}{\sqrt{x^2+y^2+(z-z')^2}} \middle| \frac{1}{\frac{1}{c}\frac{2(z-z_0)}{x\sqrt{x^2+y^2+(z-z')^2}}} \right] & (4) \\ &= \sum_{z_0} \frac{c}{(z-z_0)} & (5) \end{aligned}$$

Now, I need to find the z_0s. First, note that there are values for which the argument of the δ function cannot go to zero. I.e.

$$t - \frac{\sqrt{x^2+y^2+(z-z')^2}}{c} = 0 \tag{6}$$

cannot be satisfied if $x^2 + y^2 = \rho^2 > t^2c^2$. This is where the Θ function comes in.

Then, $z_0 = z \pm \sqrt{t^2c^2 - \rho^2}$. So the solution is

$$\psi(\vec{x},t) = \Theta(ct-\rho)\left[\frac{c}{1+\sqrt{t^2c^2-\rho^2}} + \frac{c}{1-\sqrt{t^2c^2-\rho^2}}\right] \quad (7)$$

$$= \frac{2c\Theta(ct-\rho)}{\sqrt{t^2c^2-\rho^2}} \quad (8)$$

2 Part b

Now the source term is $f(x',t') = \delta(x')\delta(t')$. I'll do this in cylindrical coordinates, and let the pulsed source be in the z direction. Then the wave is,

$$\psi(\vec{x},t) = \int \frac{\delta(z')\delta\left(t - \frac{|\vec{x}-\vec{x'}|}{c}\right)}{|\vec{x}-\vec{x'}|} d^3x' \quad (9)$$

$$= \int \rho' d\theta' d\rho' \frac{\delta\left(t - \frac{\sqrt{z^2+\rho^2+\rho'^2-2\rho\rho'\cos(\theta-\theta')}}{c}\right)}{\sqrt{z^2+\rho^2+\rho'^2-2\rho\rho'\cos(\theta-\theta')}} \quad (10)$$

Using the identity from part a to do the θ' integral,

$$\psi(\vec{x},t) = \int \rho' d\rho' \frac{1}{\sqrt{z^2+\rho^2+\rho'^2-2\rho\rho'\cos(\theta-\theta_0)}}\left[\frac{1}{\frac{-1}{2c}\frac{-\rho\rho'\sin(\theta-\theta_0)}{\sqrt{z^2+\rho^2+\rho'^2-2\rho\rho'\cos(\theta-\theta_0)}}}\right] \quad (11)$$

$$= \frac{2c}{\rho}\int \rho' d\rho' \frac{1}{\rho\rho'\sin(\theta-\theta_0)} \quad (12)$$

Where θ_0 is defined by

$$\sin(\theta-\theta_0) = \sqrt{1 - \frac{z^2+\rho^2+\rho'^2-t^2c^2}{\rho\rho'}} \quad (13)$$

Again, there are regions for which the δ function cannot be satisfied, so I have to introduce $\Theta(tc-|z|)$. So,

$$\psi(\vec{x},t) = \frac{2c}{\rho}\Theta(tc-|z|)\int \frac{2\rho\rho' d\rho'}{\sqrt{4\rho^2\rho'^2 - (z^2+\rho^2+\rho'^2-t^2c^2)^2}} \quad (14)$$

Letting $q = \rho'^2$,

$$\psi(\vec{x},t) = 2c\Theta(tc - |z|)\int_L^U \frac{dq}{\sqrt{-q^2 + q(4\rho^2 - 2(x^2+\rho^2-t^2c^2)) - (x^2+\rho^2-t^2c^2)^2}} \tag{15}$$

Which can be writttten as:

$$\psi(\vec{x},t) = 2c\Theta(tc - |z|)\int_L^U \frac{dq}{\sqrt{(q-L)(U-q)}} \tag{16}$$

Where I'll leave figuring out L and U for later (as it turns out, we don't need to know their values...just hang on). Now letting $M = \frac{U-L}{2}$ and shifting the variable from q to $q + (-M)$,

$$\psi(\vec{x},t) = 2c\Theta(tc - |z|)\int_{-M}^{+M} \frac{dq}{\sqrt{M^2 - q^2}} \tag{17}$$

Now let $q = M\sin\theta$,

$$\psi(\vec{x},t) = 2c\Theta(tc - |z|)\left[2\int_0^{\frac{\pi}{2}} d\theta \frac{\cos\theta}{\cos\theta}\right] \tag{18}$$

So, finally,

$$\psi(\vec{x},t) = 2c\,\pi\,\Theta(tc - |z|) \tag{19}$$