

1 Part a

Well, $d\vec{l}' = dz'\hat{z}$, and for L along the z-axis, $x' = y' = 0$. So,

$$\vec{x} - \vec{x}' = x\hat{x} + y\hat{y} + (z - z')\hat{z} \quad (1)$$

Then,

$$d\vec{l}'x(\vec{x} - \vec{x}') = (x\hat{y} - y\hat{x})dz' \quad (2)$$

So

$$\vec{A}(\vec{x}) = \frac{g}{4\pi} \int_L \frac{d\vec{l}'x(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|} \quad (3)$$

$$= \frac{g}{4\pi} \int_{-\infty}^0 \frac{(x\hat{y} - y\hat{x})dz'}{(x^2 + y^2 + (z - z')^2)^{\frac{3}{2}}} \quad (4)$$

Now, converting to spherical coordinates:

$$x\hat{y} - y\hat{x} = r \sin \theta \hat{\phi} \quad (5)$$

$$x^2 + y^2 + (z - z')^2 = r^2 \sin^2 \theta \cos^2 \phi + r^2 \sin^2 \theta \sin^2 \phi + (r \cos \theta + r')^2 \quad (6)$$

$$= r^2 + 2rr' \cos \theta + r'^2 \quad (7)$$

Then,

$$\vec{A}(\vec{x}) = \frac{g}{4\pi} (r \sin \theta \hat{\phi}) \int_{-\infty}^0 \frac{dr'}{(r^2 + 2rr' \cos \theta + r'^2)^{\frac{3}{2}}} \quad (8)$$

$$= \frac{g}{4\pi} (r \sin \theta \hat{\phi}) \frac{1 - \cos \theta}{r^2 \sin^2 \theta} \quad (9)$$

$$= \frac{g}{4\pi} \frac{1 - \cos \theta}{\sin \theta} \hat{\phi} \quad (10)$$

So, finally:

$$\begin{aligned} A_r &= 0 \\ A_\theta &= 0 \\ A_\phi &= \frac{g}{4\pi} \frac{1 - \cos \theta}{\sin \theta} \hat{\phi} \end{aligned} \quad (11)$$

2 Part b

$$\vec{B} = \vec{\nabla}_x \vec{A} \quad (12)$$

$$= \hat{r} \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta A_\phi) \right] - \hat{\theta} \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\phi) \right] \quad (13)$$

$$= \frac{\hat{r}}{r \sin \theta} \left[\frac{g}{4\pi r} \frac{\partial}{\partial \theta} (1 - \cos \theta) \right] - \hat{\theta} \frac{1}{r} \left[\frac{g}{4\pi} \frac{1 - \cos \theta}{\sin \theta} \frac{\partial}{\partial r} (1) \right] \quad (14)$$

$$= \frac{\hat{r}}{r \sin \theta} \left[\frac{g}{4\pi r} \sin \theta \right] \quad (15)$$

$$\vec{B}(\vec{x}) = \frac{g}{4\pi r^2} \hat{r} \quad (16)$$

So, \vec{B} falls as r^{-2} , and radiates outward from the origin and, then, is a Coulomb-like field of a point charge.

3 Part c

The total flux through the given loop will be the same as the flux through a spherical cap. So,

- $\theta < \frac{\pi}{2}$

$$\int \vec{B} \cdot \hat{n} da = \int_0^\theta d\theta \int_0^{2\pi} d\phi \frac{g}{4\pi R^2} R^2 \sin \theta \quad (17)$$

$$= \frac{g}{2} [1 - \cos \theta] \quad (18)$$

- $\theta > \frac{\pi}{2}$

Keep in mind that we are calculating the upward-going radiation, so we need to introduce a minus sign.

$$\int \vec{B} \cdot \hat{n} da = \int_\theta^\pi d\theta \int_0^{2\pi} d\phi \frac{-g}{4\pi R^2} R^2 \sin \theta \quad (19)$$

$$= \frac{-g}{2} [1 + \cos \theta] \quad (20)$$

4 Part d

$$\oint \vec{A} \cdot d\vec{l} = \oint \left(\frac{g(1 - \cos \theta)}{4\pi R \sin \theta} \hat{\phi} \right) \cdot (R \sin \theta d\phi \hat{\phi}) \quad (21)$$

$$= \frac{g}{2} (1 - \cos \theta) \quad (22)$$

Now compare this to Part c:

- $0 < \theta < \frac{\pi}{2}$

$$\int \vec{B} \cdot \hat{n} da - \oint \vec{A} \cdot d\vec{l} = \frac{g}{2}(1 - \cos \theta) - \frac{g}{2}(1 - \cos \theta) \quad (23)$$

$$= 0 \quad (24)$$

So, they are identical for a loop with $0 < \theta < \frac{\pi}{2}$.

- $\frac{\pi}{2} < \theta < \pi$

$$\int \vec{B} \cdot \hat{n} da - \oint \vec{A} \cdot d\vec{l} = \frac{-g}{2}(1 - \cos \theta) - \frac{g}{2}(1 - \cos \theta) \quad (25)$$

$$= -g \quad (26)$$

And they are different by a constant for $\frac{\pi}{2} < \theta < \pi$. This is from the flux coming up the -z-axis from the “string.”