Jackson Problem 7.12
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1 Part a

Here are the relavent equations:

- Jackson's eqn 7.58

$$\sigma(\omega) = \frac{f_0 N_e^2}{m(\gamma_0 - i\omega)} \tag{1}$$

- Ohm's Law

$$\vec{J} = \sigma \vec{E} \tag{2}$$

- continuity

$$\nabla \cdot \vec{J}(\vec{x},t) + \frac{\partial}{\partial t}\rho(\vec{x},t) = 0 \tag{3}$$

- Coulomb

$$\nabla \cdot \vec{E}(\vec{x},t) = \frac{1}{\epsilon_0}\rho(\vec{x},t) \tag{4}$$

Now, taking the time Fourier transforms of the relavent bits (using Jackson's normalization convention):

$$\vec{J}(\vec{x},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\vec{J}(\vec{x},\omega) \tag{5}$$

$$\rho(\vec{x},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\rho(\vec{x},\omega) \tag{6}$$

$$\vec{E}(\vec{x},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\vec{E}(\vec{x},\omega) \tag{7}$$

Now, I can use these in the continuity equation.

$$\nabla \cdot \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\vec{J}(\vec{x},\omega)\right] + \frac{\partial}{\partial t}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\rho(\vec{x},\omega)\right] = 0 \tag{8}$$

I can pull the ∇ and $\frac{\partial}{\partial t}$ into the integrals since the integrals are over ω, not over space or time. Then,

$$\left[\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\nabla \cdot \vec{J}(\vec{x},\omega)\right] + \left[\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\frac{\partial}{\partial t}\rho(\vec{x},\omega)\right] = 0 \tag{9}$$

Now, the transform of Ohm's law is:

$$\vec{J}(\vec{x},\omega) = \sigma(\omega)\vec{E}(\vec{x},\omega) \tag{10}$$

And, taking $\nabla\cdot$ of each side,

$$\nabla\cdot\vec{J}(\vec{x},\omega) = \sigma(\omega)\nabla\cdot\vec{E}(\vec{x},\omega) \tag{11}$$

Also, Coulomb's law transforms:

$$\nabla\cdot\vec{E}(\vec{s},\omega) = \frac{1}{\epsilon_0}\rho(\vec{x},\omega) \tag{12}$$

Plugging 11 and 12 in 9:

$$\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\frac{\sigma(\omega)}{\epsilon_0}\rho(\vec{x},\omega) - \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}(i\omega)\rho(\vec{x},\omega) = 0 \tag{13}$$

$$\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\left[\frac{\sigma(\omega)}{\epsilon_0} - i\omega\right]\rho(\vec{x},\omega) = 0 \tag{14}$$

Now, this must hold for any ρ and σ, so

$$\left[\frac{\sigma(\omega)}{\epsilon_0} - i\omega\right]\rho(\vec{x},\omega) = 0 \tag{15}$$

$$\left[\sigma(\omega) - i\omega\epsilon_0\right]\rho(\vec{x},\omega) = 0 \tag{16}$$

2 Part b

Using the given $\sigma(\omega)$ and the result from part a,

$$\left[\frac{\epsilon_0\omega_p^2\tau}{(1-i\omega\tau)\epsilon_0} - i\omega\right]\rho(\vec{x},\omega) = 0 \tag{17}$$

Now, in general $\rho(\vec{x},\omega) \neq 0$, so

$$\frac{\epsilon_0\omega_p^2\tau}{(1-i\omega\tau)\epsilon_0} - i\omega = 0 \tag{18}$$

so

$$\tau\omega^2 + i\omega - \omega_p^2\tau = 0 \tag{19}$$

Which is just a quadratic equation in ω, so

$$\omega = \frac{-i \pm \sqrt{(-1) + 4(\tau^2 \omega_p^2)}}{2\tau} \tag{20}$$

In the approximation, $\omega_p \tau >> 1$, so

$$\omega \simeq \frac{-i}{2\tau} \pm \frac{2\omega_p \tau}{2\tau} = \frac{-i}{2\tau} \pm \omega_p \tag{21}$$

Now, if we consider a plane wave with frequency ω,

$$\vec{E} = \vec{E}_0 e^{i\vec{k}\cdot\vec{x}} e^{-i\omega t} \tag{22}$$

Using the ω I just found,

$$\begin{aligned} \vec{E} &= \vec{E}_0 e^{i\vec{k}\cdot\vec{x}} e^{-i(-\frac{i}{2\tau} \pm \omega_p)t} && (23) \\ &= \left(\vec{E}_0 e^{-\frac{t}{2\tau}}\right) e^{i\vec{k}\cdot\vec{x}} e^{\mp i\omega_p t} && (24) \end{aligned}$$

And from this, it is obvious that the amplitude decays with decay constant $\frac{1}{2\tau}$, and the field oscillates with the plasma frequency.