Jackson Problem 11.3
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Let k_0 be the "lab frame," then k_1 has velocity v_1 relative to k_0, and k_2 has velocity v_2 relative to k_1. Then the transform from $k_0 \to k_1$ is:

$$A_{01} = \begin{pmatrix} \gamma_1 & v_1\gamma_1 & 0 & 0 \\ v_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Where $\gamma_1 = (1 - v_1^2)^{-\frac{1}{2}}$. And the transform from $k_1 \to k_2$ is:

$$A_{12} = \begin{pmatrix} \gamma_2 & v_2\gamma_2 & 0 & 0 \\ v_2\gamma_2 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

With $\gamma_2 = (1 - v_2^2)^{-\frac{1}{2}}$. So, transforming from $k_0 \to k_2$,

$$A_{02} = A_{12}A_{01} \tag{1}$$

$$= \begin{pmatrix} \gamma_1\gamma_2 + \gamma_1\gamma_2 v_1 v_2 & \gamma_1\gamma_2 v_2 + \gamma_1\gamma_2 v_1 & 0 & 0 \\ \gamma_1\gamma_2 v_2 + \gamma_1\gamma_2 v_1 & \gamma_1\gamma_2 + \gamma_1\gamma_2 v_1 v_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

$$= \begin{pmatrix} \gamma_1\gamma_2(1 + v_1 v_2) & \gamma_1\gamma_2(v_1 + v_2) & 0 & 0 \\ \gamma_1\gamma_2(v_1 + v_2) & \gamma_1\gamma_2(1 + v_1 v_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

Then, to make this look like a single Lorentz transform, try $\gamma = \gamma_1\gamma_2(1 + v_1 v_2)$ and $v\gamma = \gamma_1\gamma_2(v_1 + v_2)$. So,

$$v = \frac{v_1 + v_2}{1 + v_1 v_2} \tag{4}$$

So I have the right v. Now, let's see if this works for γ. I need

$$\gamma = \frac{1}{\sqrt{1 - v^2}} = \frac{1}{\sqrt{1 - \left(\frac{v_1 + v_2}{1 + v_1 v_2}\right)^2}} \tag{5}$$

Well,

$$
\begin{aligned}
\gamma &= \gamma_1\gamma_2(1+v_1v_2) && (6)\\
&= \frac{1+v_1v_2}{\sqrt{(1+v_1^2)(1-v_2^2)}} && (7)\\
&= \frac{1}{\left(\frac{1-v_2^2-v_1^2+v_1^2v_2^2}{(1+v_1v_2)^2}\right)^{\frac{1}{2}}} && (8)\\
&= \frac{1}{\left(\frac{1-v_2^2-v_1^2+v_1^2v_2^2+2v_1v_2-2v_1v_2}{(1+v_1v_2)^2}\right)^{\frac{1}{2}}} && (9)\\
&= \frac{1}{\left(\frac{(1+2v_1v_2+v_1^2v_2^2)-(v_1^2+2v_1v_2+v_2^2)}{(1+v_1v_2)^2}\right)^{\frac{1}{2}}} && (10)\\
&= \frac{1}{\left(\frac{(1+v_1v_2)^2}{(1+v_1v_2)^2}-\frac{(v_1+v_2)^2}{(1+v_1v_2)^2}\right)^{\frac{1}{2}}} && (11)\\
\gamma &= \frac{1}{\sqrt{1-\left(\frac{v_1+v_2}{1+v_1v_2}\right)^2}} && (12)
\end{aligned}
$$

Which is what I said I needed. So, two successive transforms is the same as one transform with $v = \frac{v_1+v_2}{1+v_1v_2}$.