Jackson Problem 11.14
B. J. Mattson

1 Part a

I'm going to write this once, and use it for all the parts of this:

$$\begin{aligned} A^{\alpha\beta}B_{\alpha\beta} &= A^{00}B_{00} + A^{01}B_{01} + A^{02}B_{02} + A^{03}B_{03} + A^{10}B_{10} + A^{11}B_{11} \\ &\quad + A^{12}B_{12} + A^{13}B_{13} + A^{20}B_{20} + A^{21}B_{21} + A^{22}B_{22} + A^{23}B_{23} \\ &\quad + A^{30}B_{30} + A^{31}B_{31} + A^{32}B_{32} + A^{33}B_{33} \end{aligned} \tag{1}$$

Then, using the definitions of F and $\mathcal{F}$given in the book:

$$\begin{aligned} F^{\alpha\beta}F_{\alpha\beta} &= 0 - E_x^2 - E_y^2 - E_z^2 - E_x^2 + 0 + B_z^2 + B_y^2 \\ &\quad - E_y^2 + B_z^2 + 0 + B_x^2 - E_z^2 + B_y^2 + B_x^2 + 0 && (2) \\ &= 2((B_x^2 + B_y^2 + B_z^2) - (E_x^2 + E_y^2 + E_z^2)) && (3) \\ F^{\alpha\beta}F_{\alpha\beta} &= 2(B^2 - E^2) && (4) \end{aligned}$$

$$\begin{aligned} \mathcal{F}^{\alpha\beta}F_{\alpha\beta} &= 0 - E_xB_x - E_yB_y - E_zB_z - E_xB_x + 0 - E_zB_z - E_yB_y \\ &\quad - E_yB_y - E_zB_z + 0 - E_xB_x - E_zB_z - E_yB_y - E_xB_x + 0 && (5) \\ &= -4(E_xB_x + E_yB_y + E_zB_z) && (6) \\ \mathcal{F}^{\alpha\beta}F_{\alpha\beta} &= -4(\vec{E} \cdot \vec{B}) && (7) \end{aligned}$$

$$\begin{aligned} \mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta} &= 0 - B_x^2 - B_y^2 - B_z^2 - B_x^2 + 0 + E_z^2 + E_y^2 \\ &\quad - B_y^2 + E_z^2 + 0 + E_x^2 - B_z^2 + E_y^2 + E_x^2 + 0 && (8) \\ &= 2((B_x^2 + B_y^2 + B_z^2) - (E_x^2 + E_y^2 + E_z^2)) && (9) \\ &= 2((E_x^2 + E_y^2 + E_z^2) - (B_x^2 + B_y^2 + B_z^2)) && (10) \\ \mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta} &= 2(E^2 - B^2) && (11) \end{aligned}$$

As for other invariants involving E, B? I've done the major ones involving $F^{\alpha\beta}$ and $\mathcal{F}^{\alpha\beta}$. The only other one to consider is $F^{\alpha}{}_{\alpha}F^{\beta}{}_{\beta}$...this turns out to be zero, so there are no other invariant quantities quadratic in $\vec{E}$, $\vec{B}$.

2 Part b

No, it is not possible to have an EM field where it is purely $\vec{E}$ in one frame and purely $\vec{B}$ in another. Here's the argument:

From part (a), I know that $E^2 - B^2$ and $\vec{E} \cdot \vec{B}$ are Lorentz invariant quantities. So, for two different frames:

$$E^2 - B^2 = E'^2 - B'^2 \tag{12}$$
$$\vec{E} \cdot \vec{B} = \vec{E'} \cdot \vec{B'} \tag{13}$$

Then, if in one of those frames $E = 0$,

$$-B^2 = E'^2 - B'^2 \tag{14}$$

So, to have the primed frame be purely electrical would require

$$-B^2 = E'^2 \tag{15}$$

But this is a problem since $E'^2 \geq 0$, so this condition is only met if $B = 0$ and $E' = 0$.

Now for the second part of the question. Go back to

$$E^2 - B^2 = E'^2 - B'^2 \tag{16}$$
$$\vec{E} \cdot \vec{B} = \vec{E'} \cdot \vec{B'} \tag{17}$$

Then, to have $E = 0$ in some frame,

$$-B^2 = E'^2 - B'^2 \tag{18}$$
$$0 = \vec{E'} \cdot \vec{B'} \tag{19}$$

So from the first requirement, $E'^2 - B'^2 < 0$ or $E'^2 < B'^2$. Also from the second requirement, $\vec{B'} \cdot \vec{E'} = 0$.