Jackson Problem 11.14
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1 Part a

If I can show that this equation is Ohm's law in the rest fram, since the equation is covariant, I will have shown that it is the covariant generalization of Ohm's law.

In the rest frame of the particle,

$$u'^{\alpha} = u'_{\beta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1}$$

$$J'^{\beta} = \begin{pmatrix} \rho' \\ \vec{J}' \end{pmatrix} \tag{2}$$

So,

$$u_{\beta} J^{\beta} = J'^{0} = \rho' \tag{3}$$

$$F^{\alpha\beta} u_{\beta} = F^{\alpha 0} = \begin{pmatrix} 0 \\ \vec{E}' \end{pmatrix} \tag{4}$$

Then, plugging these into the given equation,

$$\begin{pmatrix} \rho' \\ \vec{J}' \end{pmatrix} - \rho' \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} = \sigma \begin{pmatrix} 0 \\ \vec{E}' \end{pmatrix} \tag{5}$$

Equating components:

$$\rho' - \rho' = 0 \tag{6}$$

$$\vec{J}' = \sigma \vec{E}' \tag{7}$$

The second of which is just Ohm's law. So the given equation is the covariant generalization of Ohm's law.

2 Part b

From part a, I know that

$$J^{\alpha} - \left(u_{\beta} J^{\beta} \right) u^{\alpha} = \sigma F^{\alpha\beta} u_{\beta} \tag{8}$$

So,

$$J^0 - \left(u_\beta J^\beta\right) u^0 = \sigma F^{0\beta} u_\beta \tag{9}$$

$$J^i - \left(u_\beta J^\beta\right) u^i = \sigma F^{i\beta} u_\beta \tag{10}$$

Now, $j^0 = \rho$ and $u^0 = \gamma$. Then equation (9) can be written:

$$\rho - \left(u_\beta J^\beta\right) \gamma = \sigma F^{0\beta} u_\beta \tag{11}$$

Or,

$$u_\beta J^\beta = \frac{\rho - \sigma F^{0\beta} u_\beta}{\gamma} \tag{12}$$

Using this in equation(10) gives (along with $u^i = \gamma v^i$)

$$J^i - \rho v^i + \sigma F^{0\beta} u_\beta v^i = \sigma F^{i\beta} u_\beta \tag{13}$$

Let's look at a couple of these pieces by themselves.

- $\sigma F^{0\beta} u_\beta v^i$

$$F^{0\beta} u_\beta = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \end{pmatrix} \begin{pmatrix} \gamma \\ -\gamma v_x \\ -\gamma v_y \\ -\gamma v_z \end{pmatrix} = \gamma \vec{E} \cdot \vec{v} \tag{14}$$

  So $\sigma F^{0\beta} u_\beta v^i = \sigma \gamma \left(\vec{E} \cdot \vec{v}\right) v^i$

- $\sigma F^{i\beta} u_\beta$

$$F^{i\beta} u_\beta = F^{i0} u_0 + F^{ij} u_j \tag{15}$$

  Now,

$$F^{ij} = \begin{pmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_z \\ -B_y & B_x & 0 \end{pmatrix} \tag{16}$$

$$u_j = -\gamma \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \tag{17}$$

  So,

$$F^{ij} u_j = \gamma \begin{pmatrix} B_z v_y - B_y v_z \\ -B_z v_x + B_x v_z \\ B_y v_x - B_x v_y \end{pmatrix} = \gamma \left(\vec{v} \times \vec{B}\right)^i \tag{18}$$

  So $\sigma F^{i\beta} u_\beta = \sigma \gamma E^i + \sigma \gamma \left(\vec{v} \times \vec{B}\right)^i$

Using these in equation(13)

$$J^i = \sigma\gamma\left(E^i + \left(\vec{v}\times\vec{B}\right)^i - \left(\vec{E}\cdot\vec{v}\right)v^i\right) + \rho v^i \tag{19}$$

So, finally,

$$\vec{J} = \sigma\gamma\left(\vec{E} + \left(\vec{v}\times\vec{B}\right) - \left(\vec{E}\cdot\vec{v}\right)\vec{v}\right) + \rho\vec{v} \tag{20}$$

3 Part c

In the medium's rest frame,

$$J'^{\mu} = \begin{pmatrix} 0 \\ \vec{J}' \end{pmatrix} \tag{21}$$

And in the frame from part b,

$$J^{\mu} = \begin{pmatrix} \rho \\ \vec{J} \end{pmatrix} \tag{22}$$

Transforming J^{μ} into the rest frame,

$$J'^0 = \rho' = \gamma\left(\rho - \vec{v}\cdot\vec{J}\right) = 0 \tag{23}$$

Or $\rho = \vec{v}\cdot\vec{J}$. Using this in the result from part b,

$$\vec{J} = \sigma\gamma\left(\vec{E} + \vec{v}\times\vec{B} - \left(\vec{E}\cdot\vec{v}\right)\vec{v}\right) + \left(\vec{v}\cdot\vec{J}\right)\vec{v} \tag{24}$$

Assuming v is in the x-direction ($\vec{v} = v\hat{x}$),

$$\vec{J} = \sigma\gamma\left(\vec{E} + \vec{v}\times\vec{B} - \left(\vec{E}\cdot\vec{v}\right)\vec{v}\right) + J_x v^2\hat{x} \tag{25}$$

Then

$$J_x = \sigma\gamma\left(\vec{E} + \vec{v}\times\vec{B} - \left(\vec{E}\cdot\vec{v}\right)\vec{v}\right)_x + J_x v^2 \tag{26}$$

$$J_x = \gamma^2\left[\sigma\gamma\left(\vec{E} + \vec{v}\times\vec{B} - \left(\vec{E}\cdot\vec{v}\right)\vec{v}\right)\right]_x \tag{27}$$

And, finally,

$$\vec{J} = \begin{pmatrix} \sigma\gamma^3\left(\vec{E} + \vec{v}\times\vec{B} - \left(\vec{E}\cdot\vec{v}\right)\vec{v}\right)_x \\ \sigma\gamma\left(\vec{E} + \vec{v}\times\vec{B} - \left(\vec{E}\cdot\vec{v}\right)\vec{v}\right)_y \\ \sigma\gamma\left(\vec{E} + \vec{v}\times\vec{B} - \left(\vec{E}\cdot\vec{v}\right)\vec{v}\right)_z \end{pmatrix} \tag{28}$$

$$\rho = v\sigma\gamma^3\left(\vec{E} + \vec{v}\times\vec{B} - \left(\vec{E}\cdot\vec{v}\right)\vec{v}\right)_x \tag{29}$$